8104: Recitation 1

Is there any difference in outcomes, strategies, and behavior from repeating a simple game infinitely many times vs finitely many times? Deoends.

Case 1: Bertrand (1883)

We begin by considering the model of oligopolistic competition proposed by Bertrand (1883). There are two profit-maximizing firms, firms 1 and 2, in a market whose demand function is given by $x(p)$. The two firms have constant returns to scale technologies with the same cost, $c > 0$, per unit produced. We assume that $x(c) \in (0, \infty)$. Competition occurs as follows: The two firms simultaneously name their prices p_1 and p_2. Sales for firm j are then given by:

$$x_j(p_j, p_k) = \begin{cases} x(p_j) & \text{if } p_j < p_k \\ \frac{1}{2}x(p_j) & \text{if } p_j = p_k \\ 0 & \text{if } p_j > p_k. \end{cases}$$

The firms incur in production costs only for an output level equal to their actual sales. Given prices p_j and p_k, firm j 's profits are therefore equal to $\pi_{j,t} = (p_j - c)\, x_j(p_j, p_k)$.

Proposition: There is a unique Nash equilibrium (p_1^*, p_2^*) in the Bertrand duopoly model. In this equilibrium, both firms set their prices equal to cost: $p_1^* = p_2^* = c$.
Proof: Page 388 - 389 MWG.

Now, let's consider the case in which the firms compete following Bertrand (1) finitely many times and (2) infinitely many times:

1. Firms compete for sales repeatedly, with competition in each period t described by the Bertrand model. When they do so, the two firms know all the prices that have been chosen (by both firms) previously. There is a discount factor $\delta < 1$, and each firm j attempts to maximize the discounted value of profits, $\sum_{t=1}^{T} \delta^{t-1}\pi_{jt}$, where π_{jt} is firm j 's profit in period t.

   In this repeated Bertrand game, firm j 's strategy specifies what price p_{jt} it will charge in each period t as a function of the history of all past price choices by the two firms, $H_{t-1} = \{p_{1\tau}, p_{2\tau}\}_{\tau=1}^{t-1}$.

   Note: This type of strategies could allow for retaliation or cooperation! Strategies of this form allow for a range of interesting behaviors. For example, a firm's strategy could call for retaliation if the firm's rival ever lowers its price below some "threshold price." This retaliation could be brief, calling for the firm to lower its price for only a few periods after the rival "crosses the line," or it could be unrelenting. The retaliation could be tailored to the amount by which the firm's rival undercut it, or it could be severe no matter how minor the rival's transgression. The firm could also respond with increasingly cooperative

   Q: Can retaliation or cooperation arise in a Subgame Perfect Nash Equilibra (SPNE) for the model where firms compete against each other only a finite number of times?

   A: No, we always go back to prices equal to marginal cost. Why? Go to the last period (T) and repeat the analysis from the static game. The unique subgame perfect Nash equilibrium of the finitely repeated Bertrand game simply involves T repetitions of the static Bertrand equilibrium in which prices equal cost. This is a simple consequence of backward induction: In the last period, T, we must be at the Bertrand solution, and therefore profits are zero in that period regardless of what has happened earlier. But, then, in period $T - 1$ we are, strategically speaking, at the last period, and the Bertrand solution must arise again. And so on, until we get to the first period. In summary, backward induction rules out the possibility of more cooperative behavior in the finitely repeated Bertrand game.

   Remark: This reasoning will be always true $\forall T < \infty$.

2. Q: Will this change when the horizon is extendend to an infinite number of periods? Yes.

   Consider the following strategy for firms $j = 1, 2$:

$$p_{jt}(H_t, 1) = \begin{cases} p^m & \text{if all elements of } H_{t-1} \text{ equal } (p^m, p^m) \text{ or } t = 1 \\ c & \text{otherwise.} \end{cases}$$

   In words, firm j 's strategy calls for it to initially play the monopoly price p^m in period 1. Then, in each period $t > 1$, firm j plays p^m if in every previous period both firms have charged price p^m and otherwise charges a price equal to cost. This type of strategy is called a Nash reversion strategy: Firms cooperate until someone deviates, and any deviation triggers a permanent retaliation in which both firms thereafter set their prices equal to cost, the one-period Nash strategy.

   Proposition: The strategies described before constitute a subgame perfect Nash equilibrium (SPNE) of the infinitely repeated Bertrand duopoly game if and only if $\delta \geq 1/2$.
   Proof: MWG 401-402.

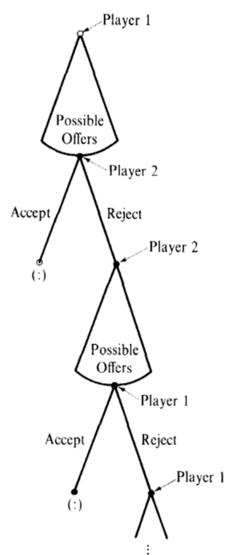
   Proposition: In the infinitely repeated Bertrand duopoly game, when $\delta \geq 1/2$ repeated choice of any price $p \in [c, p^m]$ can be supported as a subgame perfect Nash equilibrium outcome path using Nash reversion strategies. By contrast, when $\delta < 1/2$, any subgame perfect Nash equilibrium outcome path must have all sales occurring at a price equal to c in every period.
   Proof: MWG 403-404.

   Conclusion: There are same games, like the one in the 8103 final, where things can change dramatically when the horizon is extended to an infinite number of periods. Is this always the case? No.

Case 2: Rubinstein (1982)

Once again we start by studying a finite horizon model of bargaining and then consider its infinite counterpart.

Set-up: Two players, P1 and P2, bargain to determine the split of v dollars. The rules are as follows: The game begins in period 1; in period 1, player 1 makes an offer of a split (a real number between 0 and v) to player 2, which player 2 may then accept or reject. If she accepts, the proposed split is immediately implemented and the game ends. If she rejects, nothing happens until period 2. In period 2, the players' roles are reversed, with player 2 making an offer to player 1 and player 1 then being able to accept or reject it. Each player has a discount factor of $\delta \in (0, 1)$, so that a dollar received in period t is worth δ^{t-1} in period 1 dollars. However, after some finite number of periods T, if an agreement has not yet been reached, the bargaining is terminated and the players each receive nothing. The game in extensive form is given by:



<u>Proposition:</u> There is a unique subgame perfect Nash equilibrium (SPNE) in this game.

Proof: Suppose first that T is odd, so that player 1 makes the offer in period T if no previous agreement has been reached. Now, player 2 is willing to accept any offer in this period because she will get zero if she refuses and the game is terminated (she is indifferent about accepting an offer of zero). Given this fact, the unique SPNE in the subgame that begins in the final period when no agreement has been previously reached has player 1 offer player 2 zero and player 2 accept. Therefore, the payoffs from equilibrium play in this subgame are $\left(\delta^{T-1}v, 0\right)$.

Now consider playing in the subgame starting in period $T-1$ when no previous agreement has been reached. Player 2 makes the offer in this period. In any SPNE, player 1 will accept an offer in period $T-1$ if and only if it provides her with a payoff of at least $\delta^{T-1}v$, since otherwise she will do better rejecting it and waiting to make an offer in period T (she earns $\delta^{T-1}v$ by doing so). Given this fact, in any SPNE, player 2 will make an offer in period $T-1$ that gives P1 a payoff of exactly $\delta^{T-1}v$, and player 1 accepts this offer (note that this is player 2's best offer among all those that would be accepted, and making an offer that will be rejected is worse for player 2 because it results in her receiving a payoff of zero). The payoffs arising if the game reaches period $T-1$ must therefore be $\left(\delta^{T-1}v, \delta^{T-2}(v-\delta v)\right) = \left(\delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v\right)$. P2 gives δv to P1 in the $T-1$, this gives a P1 payoff of $\delta^{T-1}(\delta v)$ and the remaining dollars can be kept by P2.

Now consider playing in the subgame starting in period $T-2$ when no previous agreement has been reached. Player 1 makes the offer in this period. In any SPNE, player 2 will accept an offer in period $T-2$ if and only if it provides her with a payoff of at least $\delta^{T-2}v - \delta^{T-1}v$, since otherwise, she will do better rejecting it and waiting to make an offer in period $T-1$. Given this fact, in any SPNE, player 1 will make an offer in period $T-2$ that gives P2 a payoff of exactly $\delta^{T-2}v - \delta^{T-1}v$, and player 2 accepts this offer. P1 offers $v\delta - v\delta^2$ since at period $T-2$ this gives P2 a payoff of $\delta^{(T-2)-1}\left(\delta v - \delta^2 v\right) = \delta^{T-3}\left(\delta v - \delta^2 v\right) = \left(\delta^{T-2}v - \delta^{T-1}v\right)$. Because of this P1 is able to keep $v - \delta v + \delta^2 v$ dollars that have a payoff of $\delta^{(T-3)}\left(v - \delta v + \delta^2 v\right) = v\left[\delta^{(T-3)} - \delta^{(T-2)} + \delta^{(T-1)}\right]$

Continuing in this fashion, we can determine that the unique SPNE when T is odd results in an agreement being reached in period 1, a payoff for player 1 of:

$$\begin{aligned} v_1^*(T) &= v\left[1 - \delta + \delta^2 - \cdots + \delta^{T-1}\right] \\ &= v\left[(1-\delta)\left(\frac{1-\delta^{T-1}}{1-\delta^2}\right) + \delta^{T-1}\right], \end{aligned}$$

and a payoff to player 2 of $v_2^*(T) = v - v_1^*(T)$.
Note that as the number of periods grows large ($T \to \infty$), P1 payoff converges to $v/(1+\delta)$, and player 2's payoff converges to $\delta v/(1+\delta)$.

Q: Can we repeat the same process for the SPNE of the infinite game?

A: No, we can't. Moreover, as we have shown, a broad range of behavior might emerge once we repeat the model infinitely many times.

Q: Will this be the case with this model? No, in this model there is a unique SPNE and it turns out to be the limiting outcome of the finite horizon model as $T \to \infty$.

<u>Proposition:</u> There is a unique SPNE. In this equilibrium, the players reach an immediate agreement in period 1, with player 1 earning $v/(1+\delta)$ and player 2 earning $\delta v/(1+\delta)$.

Proof: The method of analysis we use here makes heavy use of the stationarity of the game. This means that the subgame starting in period 2 looks exactly like that in period 1, but with the players' roles reversed.

To start, let $\bar{v}_1$ denote the largest payoff that player 1 gets in any SPNE (i.e., there may, in principle, be multiple SPNEs in this model). Given the stationarity of the model, this is also the largest amount in dollars that player 2 can expect in the subgame that begins in period 2 after her rejection of player 1's period 1 offer, a subgame in which player 2 has the role of being the first player to make an offer. Then, the biggest payoff P2 can get in the subgame that begins in period 2 after her rejection of player 1's period 1 offer is $\delta\bar{v}_1$.

As a result, P1's payoff in any SPNE cannot be lower than the amount $\underline{v}_1 = v - \delta\bar{v}_1$ because, if it was, then player 1 could do better by making a period 1 offer that gives player 2 just slightly more than $\delta\vec{v}_1$. Player 2 is certain to accept any such offer because she will earn only $\delta\bar{v}_1$ by rejecting it. Again by stationarity this implies that the smallest payoff that P2 can get in the subgame that begins in period 2 after her rejection of player 1's period 1 offer is $\delta\underline{v}_1$.

Claim: In any SPNE, $\bar{v}_1$ cannot be larger than $v - \delta\underline{v}_1$.

To see this, note that in any SPNE, player 2 is certain to reject any offer in period 1 that gives her less than $\delta\underline{v}_1$ because she can earn at least $\delta\underline{v}_1$ by rejecting it and waiting to make an offer in period 2. Thus, player 1 can do no better than $v - \delta\underline{v}_1$ by making an offer that is accepted in period 1. What about by making an offer that is rejected in period 1? Since player 2 must earn at least $\delta\underline{v}_1$ if this happens, and since agreement cannot occur before period 2, player 1 can earn no more than $\delta v - \delta\underline{v}_1$ by doing this. Hence, since $\delta v - \delta\underline{v}_1 \leq v - \delta\underline{v}_1$ we know that $\bar{v}_1 \leq v - \delta\underline{v}_1$.

Note that these derivations imply that

$$\bar{v}_1 \leq v - \delta\underline{v}_1 = (\underline{v}_1 + \delta\bar{v}_1) - \delta\underline{v}_1$$

so that

$$\bar{v}_1(1-\delta) \leq \underline{v}_1(1-\delta)$$

Given the definitions of $\underline{v}_1$ and $\bar{v}_1$, this implies that $\underline{v}_1 = \bar{v}_1$, and so player l's SPNE payoff is uniquely determined. Denote this payoff by v_1°. Since $v_1^\circ = v - \delta v_1^\circ$, we find that player 1 must earn $v_1^\circ = v/(1+\delta)$ and player 2 must earn $v_2^\circ = v - v_1^\circ = \delta v/(1+\delta)$. In addition, we see that an agreement will be reached in the first period since player 1 will find it worthwhile to make an offer that player 2 accepts.

The SPNE strategies are as follows: A player who has just received an offer accepts it if and only if she is offered at least δv_1°, while a player whose turn it is to make an offer offers exactly δv_1° to the player receiving the offer.

Note that the equilibrium strategies, outcome, and payoffs are precisely the limit of those in the finite game as $T \to \infty$.

The coincidence of the infinite horizon equilibrium with the limit of the finite horizon equilibria in this model is not a general property of infinite horizon games.