Lecture 1

Games on Consequences

Defn: A game (G) of Consequences (C) is:

$I = \{1, \dots, i, \dots, n\}$ set of players

$\forall_i \quad A^i$: finite set of actions

$A \equiv \bigtimes_{i=1}^{n} A^i$ set of action profiles.

$a \in A$, so $a = (a^1, \dots a^i \dots a^n)$

C = finite set of consequences

$g : A \to C$ (game function)

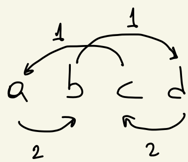
$\forall_i \ \succsim_i$ on $C \times C$ ∎

Example: $A^1 = \{T, B\}$, $A^2 = \{L, R\}$

$C = \{a, b, c, d\}$

g is

$$\begin{array}{cccc} & L & & R \\ T & a & \succ_2 & b \\ & \curlywedge_1 & & \curlyvee_1 \\ B & c & \prec_2 & d \end{array}$$



In this example GT cannot make any prediction.

So, we need to extend the theory. (vN – M)

von Neuman   Morgen stern

Decision Theory (vNM)

(1) $\Delta(C) = \{ p = (p^1, \dots, p^x, \dots, p^k) : \forall_k \; p^k \geq 0, \; \sum_k p^k = 1 \}$

$K = \# C$ = set of lotteries on C.

(2) $\forall i$, $\succsim^i$ a preference order on $\Delta(C)$

Drop the i index in $\succsim^i$

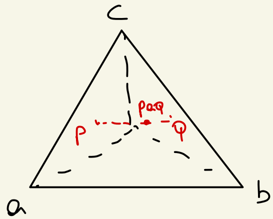
Note: $(\Delta(C), \quad M \quad)$ is a mixture space.

Defn:

A mixture space is a set $(\Delta(C))$ and $M : \Delta(C) \times \Delta(C) \times [0,1] \to \Delta(C)$

for p, q, a, $p, q \in \Delta(C)$, $a \in [0,1]$, $M(p, q, a) = paq$

[example: $paq = ap + (1-a)q \in \Delta(C)$] that satisfies $p, q, r \in \Delta(C), a, b \in [0,1]$



(1) $p1q = p$

(2) $paq = q(1-a)p$

(3) $(paq)br = pa(qbr)$

From:
$$\left[\begin{array}{ccc} & L & R \\ T & a & b \\ B & c & d \end{array}, (\succsim^1, \succsim^2)\right] \Longrightarrow$$

To:
$$\left[\begin{array}{ccc} & L & R \\ T & u^1(a), u^2(a) & u^1(b), u^2(b) \\ B & u^3(a), u^3(b) & u^4(a), u^4(b) \end{array}\right]$$

Player 1's POV: if $(q, 1-q)$ is prob that player 2.

$$\begin{array}{ccc} & Q & 1-Q \\ & L & R \\ T & u^1(a) & u^1(b) \\ B & u^3(a) & u^4(a) \end{array}$$

Utility for 1 of T is $q\,u^1(a) + (1-q)\,u^1(b)$

What we want: (1) $u^1(a)$: utility representation (2) $q\,u^1(a) + (1-q)\,u^1(b)$

ability to combine utilities.

Axioms of $\succsim$:

~~For every i~~, $\succsim^i$ satisfies: (Drop i notation)

(1): complete, transitive:

Complete: $(p \succsim q$ or $q \succsim p)$

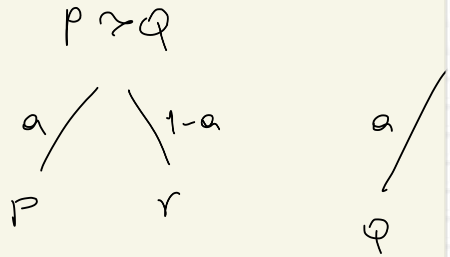
Transitive: $\forall p, q, r \; (p \succsim q \wedge q \succsim r \Rightarrow p \succsim r)$

Independence

(2): $\forall p, q \quad \forall \alpha \in (0,1)$:

$$p \succ q \Rightarrow \forall_{r \in \Delta(C)} \; p\alpha r \succ q\alpha r$$

we could define this with weak preference as in MWG.



Continuity: $\forall L_1, L_2, L_3 \in \mathcal{L}$, s.t. $L_1 \succsim L_2 \succsim L_3$, the sets

$\{\alpha \mid L_1 \alpha L_3 \succsim L_2\}$ and

$\{\beta \mid L_1 \alpha L_3 \precsim L_2\}$ are

closed.

Alternatively: $\exists\ \alpha \in [0,1]$ s.t.

$L_1 \alpha L_3 \sim L_2$.

Continuity

(3): $\forall\ P, Q, r \in \Delta(C)$

$$P \succ Q \succ r \Rightarrow \exists_{a,b \in (0,1)}\ \ Par \succ Q \succ pbr$$

(it already holds for $a=1$, $b=0$. This axiom says we can find $a, b \in (0,1)$ as well.)

"Axioms (1),(2),(3) are satisfied iff there exists an expected utility representation of $\succsim$ and such representation is essentially unique in the class of EUR"
(exp uti rep)

Defn: $U: \Delta(C) \to \mathbb{R}$ is a representation of $\succsim$ iff

$$\forall_{P,Q}\ (P \succsim Q \text{ iff } U(P) \geq U(Q))$$

Defn: U is an expected utility representation of $\succsim$ iff

(1) U is a representation.

(2) $\exists\, u: C \to \mathbb{R}$ s.t. $\forall_p\ U(P) = \sum_{c \in C} P(c)\, u(c)$ ■

Probability of consequences.

Theorem (VNM):

(1) (existence): $\succsim$ satisfies A 1,2,3 iff there exists an EUR U of $\succsim$,

(2) (uniqueness): If U is an EUR of $\succsim$

then $\left(V \text{ is EUR of } \succsim \iff \exists_{a>0,\, b}\ V = aU + b \right)$

MLT

Monotonic Linear transformation.

Proof:

(2) uniqueness proof: Let U be EUR.

($\Leftarrow$) claim $aU+b = V$ is EUR if $a>0$.

(R): $P \succsim Q \implies U(P) \geq U(Q)$

$\implies aU(P)+b \geq aU(Q)+b \quad (a>0,\ b\geq 0)$

$\implies V(P) \geq V(b)$

(EU): Claim: V is EU.

$\exists\, u : C \to \mathbb{R} \quad \forall p \quad U(p) = \sum p(c)\, u(c)$ (we know this)

We want

$$\exists\, v: C \to \mathbb{R} \quad \forall p \quad V(p) = \sum_c p(c)\, v(c)$$

Let $v(c) = a\,u(c) + b$

$$\sum_c p(c)\, v(c) = \sum_c p(c)\left(a\,u(c) + b\right)$$

$$= a \sum p(c)\, u(c) + b \sum p(c)$$

$$= a\, U(p) + b = V(p).$$

$(\Longrightarrow)$ V EUR of $\succsim \implies \exists_{a>0,b}\ V = aU + b$

Let $p \in \Delta(C)$.

$$\Big(\succsim \text{ on } \Delta(C),\ (0 \dots \underset{k\,th}{1}, \dots 0) \in \Delta(C) \text{ is } c^k \quad \succsim\big|_C \Big)$$

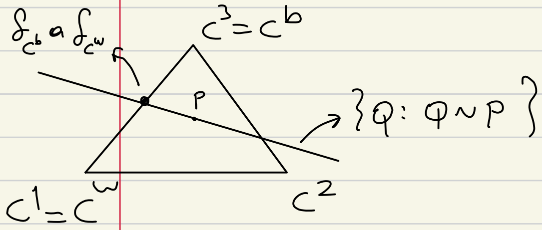
Define $c_b = (\text{best Consequence})$ s.t. $\forall c \in C \quad c_b \succsim\big|_C\, c$

$c_w = (\text{worst Consequence})$ s.t. $\forall c \in C \quad c_w \precsim\big|_C\, c$

WLOG $c_b \succ c_w$ ($c_b \approx c_w$ is trivial. We do nothing in this case.)

The axioms imply (prove it!)

$\exists !\, a \in [0,1]$ $\quad P \sim \delta_{c_b}\, a\, \delta_{c_w}$ where $\delta_{c^k} \equiv (0, \dots, \overset{k\text{th}}{1}, 0, 0 \dots 0)$



Since both U and V are EUR,

$V(P) = V(\delta_{c_b}\, a\, \delta_{c^w})$ (since V is R) $\to$ vnm representation $\quad V(\delta_{c_b}\, a\, \delta_{c^w}) = a\, V(\delta_{cb}) + (1-a)\, V(\delta_{c^w})$

$V(P) = a\, V(c^b) + (1-a)\, V(c^w)$

$U(P) = a\, U(c^b) + (1-a)\, U(c^w)$

We now pose the decision maker's choice problem in the general framework developed in Chapter 1 (see Section 1.B). In accordance with our consequentialist premise, we take the set of alternatives, denoted here by $\mathscr{L}$, to be *the set of all simple lotteries over the set of outcomes C*. We next assume that the decision maker has a rational preference relation $\succsim$ on $\mathscr{L}$, a complete and transitive relation allowing comparison of any pair of simple lotteries. It should be emphasized that, if anything, the rationality assumption is stronger here than in the theory of choice under certainty discussed in Chapter 1. The more complex the alternatives, the heavier the burden carried by the rationality postulates. In fact, their realism in an uncertainty context has been much debated. However, because we want to concentrate on the properties that are specific to uncertainty, we do not question the rationality assumption further here.

We next introduce two additional assumptions about the decision maker's preferences over lotteries. The most important and controversial is the *independence axiom*. The first, however, is a continuity axiom similar to the one discussed in Section 3.C.

Definition 6.B.3: The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ is *continuous* if for any $L, L', L'' \in \mathscr{L}$, the sets

$$\{\alpha \in [0,1]: \alpha L + (1-\alpha)L' \succsim L''\} \subset [0,1]$$

and

$$\{\alpha \in [0,1]: L'' \succsim \alpha L + (1-\alpha)L'\} \subset [0,1]$$

are closed.

In words, continuity means that small changes in probabilities do not change the nature of the ordering between two lotteries. For example, if a "beautiful and uneventful trip by car" is preferred to "staying home," then a mixture of the outcome "beautiful and uneventful trip by car" with a sufficiently small but positive probability of "death by car accident" is still preferred to "staying home." Continuity therefore rules out the case where the decision maker has lexicographic ("safety first") preferences for alternatives with a zero probability of some outcome (in this case, "death by car accident").

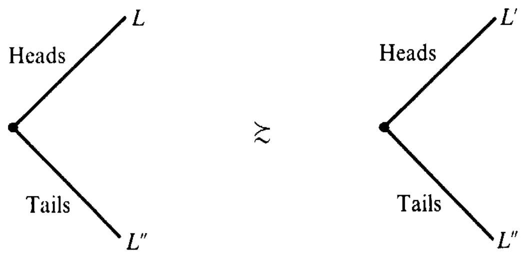
As in Chapter 3, the continuity axiom implies the existence of a utility function representing $\succsim$, a function $U: \mathscr{L} \to \mathbb{R}$ such that $L \succsim L'$ if and only if $U(L) \geq U(L')$. Our second assumption, the independence axiom, will allow us to impose considerably more structure on $U(\cdot)$.[5]

Definition 6.B.4: The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ satisfies the *independence axiom* if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0,1)$ we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

In other words, if we mix each of two lotteries with a third one, then the preference ordering of the two resulting mixtures does not depend on (is *independent* of) the particular third lottery used.

5. The independence axiom was first proposed by von Neumann and Morgenstern (1944) as an incidental result in the theory of games.



if and only if

$L \succsim L'$

Figure 6.B.4
The independence axiom.

Suppose, for example, that $L \succsim L'$ and $\alpha = \frac{1}{2}$. Then $\frac{1}{2}L + \frac{1}{2}L''$ can be thought of as the compound lottery arising from a coin toss in which the decision maker gets L if heads comes up and L'' if tails does. Similarly, $\frac{1}{2}L' + \frac{1}{2}L''$ would be the coin toss where heads results in L' and tails results in L'' (see Figure 6.B.4). Note that conditional on heads, lottery $\frac{1}{2}L + \frac{1}{2}L''$ is at least as good as lottery $\frac{1}{2}L' + \frac{1}{2}L''$; but conditional on tails, the two compound lotteries give identical results. The independence axiom requires the sensible conclusion that $\frac{1}{2}L + \frac{1}{2}L''$ be at least as good as $\frac{1}{2}L' + \frac{1}{2}L''$.

The independence axiom is at the heart of the theory of choice under uncertainty. It is unlike anything encountered in the formal theory of preference-based choice discussed in Chapter 1 or its applications in Chapters 3 to 5. This is so precisely because it exploits, in a fundamental manner, the structure of uncertainty present in the model. In the theory of consumer demand, for example, there is no reason to believe that a consumer's preferences over various bundles of goods 1 and 2 should be independent of the quantities of the other goods that he will consume. In the present context, however, it *is* natural to think that a decision maker's preference between two lotteries, say L and L', should determine which of the two he prefers to have as part of a compound lottery *regardless* of the other possible outcome of this compound lottery, say L''. This other outcome L'' should be irrelevant to his choice because, in contrast with the consumer context, he does not consume L or L' together with L'' but, rather, only *instead* of it (if L or L' is the realized outcome).

Exercise 6.B.1: Show that if the preferences $\succsim$ over $\mathscr{L}$ satisfy the independence axiom, then for all $\alpha \in (0, 1)$ and $L, L', L'' \in \mathscr{L}$ we have

$$L \succ L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''$$

and

$$L \sim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''.$$

Show also that if $L \succ L'$ and $L'' \succ L'''$, then $\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'''$.

As we will see shortly, the independence axiom is intimately linked to the representability of preferences over lotteries by a utility function that has an *expected utility form.* Before obtaining that result, we define this property and study some of its features.

CHAPTER 6

6.B.1 Suppose first that $L \succ L'$. A first application of the independence axiom (in the "only-if" direction in Definition 6.B.4) yields

$$\alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''.$$

If these two compound lotteries were indifferent, then a second application of the independence axiom (in the "if" direction) would yield $L' \succsim L$, which contradicts $L \succ L'$. We must thus have

$$\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''.$$

Suppose conversely that $\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''$, then, by the independence axiom, $L \succsim L'$. If these two simple lotteries were indifferent, then the independence axiom would imply

$$\alpha L' + (1 - \alpha)L'' \succsim \alpha L + (1 - \alpha)L'',$$

a contradiction. We must thus have $L \succ L'$.

Suppose next that $L \sim L'$, then $L \succsim L'$ and $L' \succsim L$. Hence by applying the independence axiom twice (in the "only if" direction), we obtain

$$\alpha L + (1 - \alpha)L'' \sim \alpha L' + (1 - \alpha)L''.$$

Conversely, we can show that if $\alpha L + (1 - \alpha)L'' \sim \alpha L' + (1 - \alpha)L''$, then $L \sim L'$.

For the last part of the exercise, suppose that $L \succ L'$ and $L'' \succ L'''$, then, by the independence axiom and the first assertion of this exercise,

$$\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''$$

and

$$\alpha L' + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L'''.$$

Thus, by the transitivity of $\succ$ (Proposition 1.B.1(i)),

$$\alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L'''.$$

Definition 6.B.5: The utility function $U: \mathscr{L} \to \mathbb{R}$ has an *expected utility form* if there is an assignment of numbers $(u_1, \ldots, u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, \ldots, p_N) \in \mathscr{L}$ we have

$$U(L) = u_1 p_1 + \cdots + u_N p_N.$$

A utility function $U: \mathscr{L} \to \mathbb{R}$ with the expected utility form is called a *von Neumann–Morgenstern (v.N–M) expected utility function*.

Observe that if we let L^n denote the lottery that yields outcome n with probability one, then $U(L^n) = u_n$. Thus, the term *expected utility* is appropriate because with the v.N–M expected utility form, the utility of a lottery can be thought of as the expected value of the utilities u_n of the N outcomes.

The expression $U(L) = \sum_n u_n p_n$ is a general form for a *linear function in the probabilities* $(p_1, \ldots, p_N)$. This linearity property suggests a useful way to think about the expected utility form.

Proposition 6.B.1: A utility function $U: \mathscr{L} \to \mathbb{R}$ has an expected utility form if and only if it is *linear*, that is, if and only if it satisfies the property that

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k) \tag{6.B.1}$$

for any K lotteries $L_k \in \mathscr{L}$, $k = 1, \ldots, K$, and probabilities $(\alpha_1, \ldots, \alpha_K) \geq 0$, $\sum_k \alpha_k = 1$.

Proof: Suppose that $U(\cdot)$ satisfies property (6.B.1). We can write any $L = (p_1, \ldots, p_N)$ as a convex combination of the degenerate lotteries $(L^1, \ldots, L^N)$, that is, $L = \sum_n p_n L^n$. We have then $U(L) = U(\sum_n p_n L^n) = \sum_n p_n U(L^n) = \sum_n p_n u_n$. Thus, $U(\cdot)$ has the expected utility form.

In the other direction, suppose that $U(\cdot)$ has the expected utility form, and consider any compound lottery $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$, where $L_k = (p_1^k, \ldots, p_N^k)$. Its reduced lottery is $L' = \sum_k \alpha_k L_k$. Hence,

$$U\left(\sum_k \alpha_k L_k\right) = \sum_n u_n \left(\sum_k \alpha_k p_n^k\right) = \sum_k \alpha_k \left(\sum_n u_n p_n^k\right) = \sum_k \alpha_k U(L_k).$$

Thus, property (6.B.1) is satisfied. ∎

The expected utility property is a *cardinal* property of utility functions defined on the space of lotteries. In particular, the result in Proposition 6.B.2 shows that the expected utility form is preserved only by increasing *linear* transformations.

Proposition 6.B.2: Suppose that $U: \mathscr{L} \to \mathbb{R}$ is a v.N–M expected utility function for the preference relation $\succsim$ on $\mathscr{L}$. Then $\tilde{U}: \mathscr{L} \to \mathbb{R}$ is another v.N–M utility function for $\succsim$ if and only if there are scalars $\beta > 0$ and γ such that $\tilde{U}(L) = \beta U(L) + \gamma$ for every $L \in \mathscr{L}$.

Proof: Begin by choosing two lotteries $\bar{L}$ and $\underline{L}$ with the property that $\bar{L} \succsim L \succsim \underline{L}$ for all $L \in \mathscr{L}$.[6] If $\bar{L} \sim \underline{L}$, then every utility function is a constant and the result follows immediately. Therefore, we assume from now on that $\bar{L} \succ \underline{L}$.

6. These best and worst lotteries can be shown to exist. We could, for example, choose a maximizer and a minimizer of the linear, hence continuous, function $U(\cdot)$ on the simplex of probabilities, a compact set.

Note first that if $U(\cdot)$ is a v.N–M expected utility function and $\tilde{U}(L) = \beta U(L) + \gamma$, then

$$\begin{aligned}
\tilde{U}\left(\sum_{k=1}^{K} \alpha_k L_k\right) &= \beta U\left(\sum_{k=1}^{K} \alpha_k L_k\right) + \gamma \\
&= \beta\left[\sum_{k=1}^{K} \alpha_k U(L_k)\right] + \gamma \\
&= \sum_{k=1}^{K} \alpha_k [\beta U(L_k) + \gamma] \\
&= \sum_{k=1}^{K} \alpha_k \tilde{U}(L_k).
\end{aligned}$$

Since $\tilde{U}(\cdot)$ satisfies property (6.B.1), it has the expected utility form.

For the reverse direction, we want to show that if both $\tilde{U}(\cdot)$ and $U(\cdot)$ have the expected utility form, then constants $\beta > 0$ and γ exist such that $\tilde{U}(L) = \beta U(L) + \gamma$ for all $L \in \mathscr{L}$. To do so, consider any lottery $L \in \mathscr{L}$, and define $\lambda_L \in [0, 1]$ by

$$U(L) = \lambda_L U(\bar{L}) + (1 - \lambda_L) U(\underline{L}).$$

Thus

$$\lambda_L = \frac{U(L) - U(\underline{L})}{U(\bar{L}) - U(\underline{L})} \tag{6.B.2}$$

Since $\lambda_L U(\bar{L}) + (1 - \lambda_L) U(\underline{L}) = U(\lambda_L \bar{L} + (1 - \lambda_L)\underline{L})$ and $U(\cdot)$ represents the preferences $\succsim$, it must be that $L \sim \lambda_L \bar{L} + (1 - \lambda_L)\underline{L}$. But if so, then since $\tilde{U}(\cdot)$ is also linear and represents these same preferences, we have

$$\begin{aligned}
\tilde{U}(L) &= \tilde{U}(\lambda_L \bar{L} + (1 - \lambda_L)\underline{L}) \\
&= \lambda_L \tilde{U}(\bar{L}) + (1 - \lambda_L)\tilde{U}(\underline{L}) \\
&= \lambda_L (\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})) + \tilde{U}(\underline{L}).
\end{aligned}$$

Substituting for λ_L from (6.B.2) and rearranging terms yields the conclusion that $\tilde{U}(L) = \beta U(L) + \gamma$, where

$$\beta = \frac{\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})}{U(\bar{L}) - U(\underline{L})}$$

and

$$\gamma = \tilde{U}(\underline{L}) - U(\underline{L}) \frac{\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})}{U(\bar{L}) - U(\underline{L})}.$$

This completes the proof ■

A consequence of Proposition 6.B.2 is that for a utility function with the expected utility form, differences of utilities have meaning. For example, if there are four outcomes, the statement "the difference in utility between outcomes 1 and 2 is greater than the difference between outcomes 3 and 4," $u_1 - u_2 > u_3 - u_4$, is equivalent to

$$\tfrac{1}{2}u_1 + \tfrac{1}{2}u_4 > \tfrac{1}{2}u_2 + \tfrac{1}{2}u_3.$$

Therefore, the statement means that the lottery $L = (\frac{1}{2}, 0, 0, \frac{1}{2})$ is preferred to the lottery $L' = (0, \frac{1}{2}, \frac{1}{2}, 0)$. This ranking of utility differences is preserved by all linear transformations of the v.N–M expected utility function.

Note that if a preference relation $\succsim$ on $\mathscr{L}$ is representable by a utility function $U(\cdot)$ that has the expected utility form, then since a linear utility function is continuous, it follows that $\succsim$ is continuous on $\mathscr{L}$. More importantly, the preference relation $\succsim$ must also satisfy the independence axiom. You are asked to show this in Exercise 6.B.2.

Exercise 6.B.2: Show that if the preference relation $\succsim$ on $\mathscr{L}$ is represented by a utility function $U(\cdot)$ that has the expected utility form, then $\succsim$ satisfies the independence axiom.

The expected utility theorem, the central result of this section, tells us that the converse is also true.

The Expected Utility Theorem

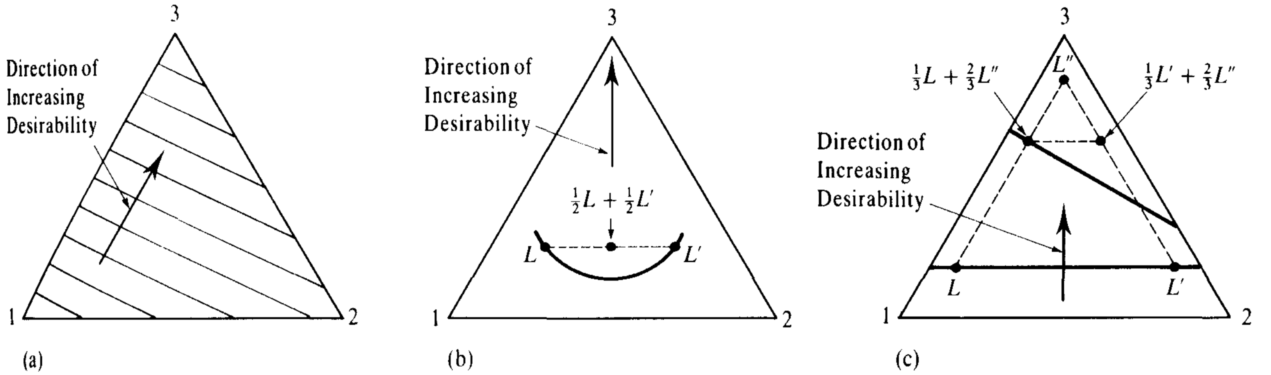
The *expected utility theorem* says that if the decision maker's preferences over lotteries satisfy the continuity and independence axioms, then his preferences are representable by a utility function with the expected utility form. It is the most important result in the theory of choice under uncertainty, and the rest of the book bears witness to its usefulness.

Before stating and proving the result formally, however, it may be helpful to attempt an intuitive understanding of why it is true.

Consider the case where there are only three outcomes. As we have already observed, the continuity axiom insures that preferences on lotteries can be represented by some utility function. Suppose that we represent the indifference map in the simplex, as in Figure 6.B.5. Assume, for simplicity, that we have a conventional map with one-dimensional indifference curves. Because the expected utility form is linear in the probabilities, representability by the expected utility form is equivalent to these indifference curves being straight, parallel lines (you should check this). Figure 6.B.5(a) exhibits an indifference map satisfying these properties. We now argue that these properties are, in fact, consequences of the independence axiom.

Indifference curves are straight lines if, for every pair of lotteries L, L', we have that $L \sim L'$ implies $\alpha L + (1-\alpha)L' \sim L$ for all $\alpha \in [0,1]$. Figure 6.B.5(b) depicts a situation where the indifference curve is not a straight line; we have $L' \sim L$ but

Figure 6.B.5 Geometric explanation of the expected utility theorem. (a) $\succsim$ is representable by a utility function with the expected utility form. (b) Contradiction of the independence axiom. (c) Contradiction of the independence axiom.



$\frac{1}{2}L' + \frac{1}{2}L \succ L$. This is equivalent to saying that

$$\tfrac{1}{2}L' + \tfrac{1}{2}L \succ \tfrac{1}{2}L + \tfrac{1}{2}L. \tag{6.B.3}$$

But since $L \sim L'$, the independence axiom implies that we must have $\frac{1}{2}L' + \frac{1}{2}L \sim \frac{1}{2}L + \frac{1}{2}L$ (see Exercise 6.B.1). This contradicts (6.B.3), and so we must conclude that indifference curves are straight lines.

Figure 6.B.5(c) depicts two straight but nonparallel indifference lines. A violation of the independence axiom can be constructed in this case, as indicated in the figure. There we have $L \succsim L'$ (in fact, $L \sim L'$), but $\frac{1}{3}L + \frac{2}{3}L'' \succsim \frac{1}{3}L' + \frac{2}{3}L''$ does not hold for the lottery L'' shown in the figure. Thus, indifference curves must be parallel, straight lines if preferences satisfy the independence axiom.

In Proposition 6.B.3, we formally state and prove the expected utility theorem.

Proposition 6.B.3: *(Expected Utility Theorem)* Suppose that the rational preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, \dots, N$ in such a manner that for any two lotteries $L = (p_1, \dots, p_N)$ and $L' = (p'_1, \dots, p'_N)$, we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_{n=1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n. \tag{6.B.4}$$

Proof: We organize the proof in a succession of steps. For simplicity, we assume that there are best and worst lotteries in $\mathscr{L}$, $\bar{L}$ and $\underline{L}$ (so, $\bar{L} \succsim L \succsim \underline{L}$ for any $L \in \mathscr{L}$).[7] If $\bar{L} \sim \underline{L}$, then all lotteries in $\mathscr{L}$ are indifferent and the conclusion of the proposition holds trivially. Hence, from now on, we assume that $\bar{L} \succ \underline{L}$.

Step 1. If $L \succ L'$ and $\alpha \in (0, 1)$, then $L \succ \alpha L + (1 - \alpha)L' \succ L'$.

This claim makes sense. A nondegenerate mixture of two lotteries will hold a preference position strictly intermediate between the positions of the two lotteries. Formally, the claim follows from the independence axiom. In particular, since $L \succ L'$, the independence axiom implies that (recall Exercise 6.B.1)

$$L = \alpha L + (1 - \alpha)L \succ \alpha L + (1 - \alpha)L' \succ \alpha L' + (1 - \alpha)L' = L'.$$

Step 2. Let $\alpha, \beta \in [0, 1]$. Then $\beta\bar{L} + (1 - \beta)\underline{L} \succ \alpha\bar{L} + (1 - \alpha)\underline{L}$ if and only if $\beta > \alpha$.

Suppose that $\beta > \alpha$. Note first that we can write

$$\beta\bar{L} + (1 - \beta)\underline{L} = \gamma\bar{L} + (1 - \gamma)[\alpha\bar{L} + (1 - \alpha)\underline{L}],$$

where $\gamma = [(\beta - \alpha)/(1 - \alpha)] \in (0, 1]$. By Step 1, we know that $\bar{L} \succ \alpha\bar{L} + (1 - \alpha)\underline{L}$. Applying Step 1 again, this implies that $\gamma\bar{L} + (1 - \gamma)(\alpha\bar{L} + (1 - \alpha)\underline{L}) \succ \alpha\bar{L} + (1 - \alpha)\underline{L}$, and so we conclude that $\beta\bar{L} + (1 - \beta)\underline{L} \succ \alpha\bar{L} + (1 - \alpha)\underline{L}$.

For the converse, suppose that $\beta \leq \alpha$. If $\beta = \alpha$, we must have $\beta\bar{L} + (1 - \beta)\underline{L} \sim \alpha\bar{L} + (1 - \alpha)\underline{L}$. So suppose that $\beta < \alpha$. By the argument proved in the previous

7. In fact, with our assumption of a finite set of outcomes, this can be established as a consequence of the independence axiom (see Exercise 6.B.3).

paragraph (reversing the roles of α and β), we must then have $\alpha\bar{L} + (1-\alpha)\underline{L} \succ \beta\bar{L} + (1-\beta)\underline{L}$.

Step 3. For any $L \in \mathscr{L}$, there is a unique α_L such that $[\alpha_L\bar{L} + (1-\alpha_L)\underline{L}] \sim L$.

Follows from continuity of $\succsim_i$ (Bipuls definition)

Existence of such an α_L is implied by the continuity of $\succsim$ and the fact that $\bar{L}$ and $\underline{L}$ are, respectively, the best and the worst lottery. Uniqueness follows from the result of Step 2.

The existence of α_L is established in a manner similar to that used in the proof of Proposition 3.C.1. Specifically, define the sets

$$\{\alpha \in [0,1]: \alpha\bar{L} + (1-\alpha)\underline{L} \succsim L\} \quad \text{and} \quad \{\alpha \in [0,1]: L \succsim \alpha\bar{L} + (1-\alpha)\underline{L}\}.$$

By the continuity and completeness of $\succsim$, both sets are closed, and any $\alpha \in [0,1]$ belongs to at least one of the two sets. Since both sets are nonempty and $[0,1]$ is connected, it follows that there is some α belonging to both. This establishes the existence of an α_L such that $\alpha_L\bar{L} + (1-\alpha_L)\underline{L} \sim L$.

Step 4. The function $U: \mathscr{L} \to \mathbb{R}$ that assigns $U(L) = \alpha_L$ for all $L \in \mathscr{L}$ represents the preference relation $\succsim$.

Observe that, by Step 3, for any two lotteries $L, L' \in \mathscr{L}$, we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha_L\bar{L} + (1-\alpha_L)\underline{L} \succsim \alpha_{L'}\bar{L} + (1-\alpha_{L'})\underline{L}.$$

Thus, by Step 2, $L \succsim L'$ if and only if $\alpha_L \geq \alpha_{L'}$.

Step 5. The utility function $U(\cdot)$ that assigns $U(L) = \alpha_L$ for all $L \in \mathscr{L}$ is linear and therefore has the expected utility form.

We want to show that for any $L, L' \in \mathscr{L}$, and $\beta \in [0,1]$, we have $U(\beta L + (1-\beta)L') = \beta U(L) + (1-\beta)U(L')$. By definition, we have

$$L \sim U(L)\bar{L} + (1-U(L))\underline{L}$$

and

$$L' \sim U(L')\bar{L} + (1-U(L'))\underline{L}.$$

Therefore, by the independence axiom (applied twice),

$$\begin{aligned}\beta L + (1-\beta)L' &\sim \beta[U(L)\bar{L} + (1-U(L))\underline{L}] + (1-\beta)L' \\ &\sim \beta[U(L)\bar{L} + (1-U(L))\underline{L}] + (1-\beta)[U(L')\bar{L} + (1-U(L'))\underline{L}].\end{aligned}$$

Rearranging terms, we see that the last lottery is algebraically identical to the lottery

$$[\beta U(L) + (1-\beta)U(L')]\bar{L} + [1 - \beta U(L) - (1-\beta)U(L')]\underline{L}.$$

In other words, the compound lottery that gives lottery $[U(L)\bar{L} + (1-U(L))\underline{L}]$ with probability β and lottery $[U(L')\bar{L} + (1-U(L'))\underline{L}]$ with probability $(1-\beta)$ has the same reduced lottery as the compound lottery that gives lottery $\bar{L}$ with probability $[\beta U(L) + (1-\beta)U(L')]$ and lottery $\underline{L}$ with probability $[1 - \beta U(L) - (1-\beta)U(L')]$. Thus

$$\beta L + (1-\beta)L' \sim [\beta U(L) + (1-\beta)U(L')]\bar{L} + [1 - \beta U(L) - (1-\beta)U(L')]\underline{L}.$$

By the construction of $U(\cdot)$ in Step 4, we therefore have

$$U(\beta L + (1 - \beta)L') = \beta U(L) + (1 - \beta)U(L'),$$

as we wanted.

20-01

Lecture 2

If U, V are EUR of $\succsim$, then they are a monotonic linear trans of each other.

U, V EUR $\Longrightarrow \exists_{a>0, b}$ st $V = aU + b$

Proof: Let c_b and c_w be the best and worst consequences.
Take $p \in \Delta(C)$, find $a \in (0,1)$ st $p \sim \delta_{c_b}\, a\, \delta_{c_w}$. Since U, V are EUR,
$U(p) \overset{①}{=} U(\delta_{c_b}\, a\, \delta_{c_w})$ and $V(p) \overset{②}{=} V(\delta_{c_b}$ and $\delta_{c_w})$.

From ① $U(p) = a\, U(\delta_{c^b}) + (1-a)\, U(\delta_{c^w})$

$= a\, u(c^b) + (1-a)\, u(c^w)$

$= a\, [u(c^b) - u(c^w)] + u(c^w)$

$V(p) = a\, v(c^b) + (1-a)\, v(c^w)$

$= v(c^w) + a\, (v(c^b) - v(c^w))$

Then $\forall p$: $\dfrac{U(p) + u(c^w)}{u(c^b) - u(c^w)} = \dfrac{V(p) + v(c^w)}{v(c^b) + v(c^w)}$

we want: $V(p) = A\, U(p) + B$. Therefore $A = v(c^b) - v(c^w) / u(c^b) - u(c^w)$.

Note: If U is a representation of $\succsim$, U^3 is a representation of $\succsim$.

VNM representation theorem:

$\boxed{\text{Now uniqness is proved: } U \text{ is a EUR then } V \text{ EUR iff } \exists_{a>0,b}\ V = AU + B.}$

$\boxed{\text{Existence: } \succsim \text{ satisfies Assumptions A.1,2,3 iff } \exists\, U \text{ EUR.}}$

$\Leftarrow$ if U EUR represents $\succsim$ then A.1 is satisfied: given p, q, Either $u(p) \geq u(q)$ then $p \succsim q$. (check why a representation does not satisfy Assumptions)

Transitivity: claim $p \succsim q \wedge q \succsim r$. Then since u is EUR $u(p) \geq u(q)$ and $u(q) \geq u(r)$ then $u(p) \geq u(r)$.

Let $\succsim$ satisfy A.1,2,3.

claim: reconstruct u EUR

step1: $c^w < c^2 < c^3 \ldots < c^k = c^b$. (If $c^2 \sim c^3$ you treat them as identical)

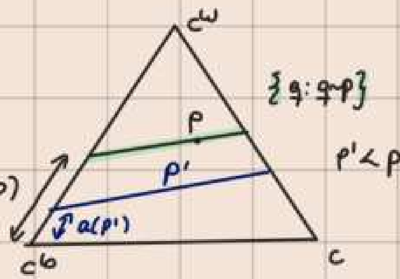
step2: $u(c^w) = 0$, $u(c^b) = 1$. We now have to find a u that satisfies this.

$\tilde{u}(c^w) = 3$ $\tilde{u}(c^b) = 17$. substract 3 and divide by 17.

step3: For any $p \in \Delta(C)$ $\exists!\, a(p) \in [0,1]$ $p \sim \delta_{c^b}\, a\, \delta_{c^w}$. $\left.\begin{matrix} \geq p \\ \delta_{c^b} \times \delta_{c^w} \leq p \end{matrix}\right\}$ and then use axioms.

step4: Define u to be equal to a $u(p) = a(p)$ and that's the EUR.

geometric Interpretation:



The lines are straight, you are indiferent between the mixture of two indifferent lotteries.

Normal Form Games: u vnm utility U utility on consequences

A game of consequences is $(I, (A^i)_{i \in I}, C, g)$. A vector of vNM preferences $(\succsim_i\ i \in I)$ - that satisfy A.1.2.3 $\Leftrightarrow$ has an EUR).

Pick any vector of representation $(u^i : i \in I)$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "(u^1, \ldots, u^n) = u"'] & C \arrow[d, "U = (U^1, \ldots, U^n)"] \\ & \mathbb{R} \end{tikzcd}$$

Note that A is a set of Actions profiles (eg (T, L)).

where $u^i(a) = U^i(g(c))$

Now rather than a consequence we have the utility of consequences:

	L	R
T	a	b
B	c	b

	L	R
T	1, -1	-1, 1
B	-1, 1	1, -1

(Matching pennies)

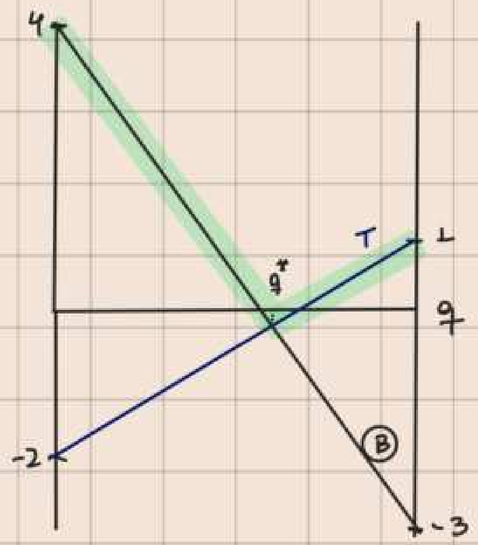
There is no Equilibrium in pure strategy.

Definition: A game in normal form is $\left(I, (A^i)_{i\in I}, (u^i)_{i\in I}\right)$ where for all i: $u^i : A \to \mathbb{R}$ and u^i is an EUR of $\succsim_i$.

<u>Mixed Extensions:</u>

	q	$1-q$
	L	R
T	1	-2
B	-3	4

What does Agent 1 prefer given Agent 2 play L with q and R with $1-q$.



$T \to q \cdot 1 + (1-q)(-2)$

$B \to -q \cdot 3 + (1-q)(4)$

If player 2 plays $q = q^*$ player 1 can choose any probability on $\{T, B\}$ the set of his own options.

Definition: A mixed strategy for player i is a $s^i \in \Delta(A^i)$.

		$s^2(L)$	$s^2(R)$
		L	R
$s^1(T)$	T	$s^1(T)s^2(L)$	$s^1(T)s^2(R)$
$s^1(B)$	B	$s^1(B)s^2(L)$	$s^1(B)s^2(R)$

The Action profile: $\text{prob}(T, L) = s^1(T)s^2(L)$

$(T, L) \to s^1(a^1)\, s^2(a^2) = \prod_{i=1}^{n} s^i(a_i)$.

Extension to many players.

Definition: For $s = (s^1, \dots, s^n)$ strategy profile $Pr_s \in \Delta(A)$ defined by for $a \in A$: $Pr_s(a) = \prod_{i=1}^{n} s^i(a_i)$.

$$s^1 = (0.9, 0.1) \quad s^2 = (0.8, 0.2) \quad A = \{ \overset{a_1}{(T,R)}, \overset{a_2}{(T,L)}, \overset{a_3}{(B,R)}, \overset{a_4}{(B,L)} \} \quad Pr_s(a) = \prod_{i=1}^{n} s^i(a^i) \quad \square$$

25-01 Normal Form Games **Lecture 3**

Mixed strategies: $S^i = \Delta(A^i)$

$(I, (A^i)_{i \in I}, (u^i)_{i \in I})$ NFG

$u^i : A \longrightarrow \mathbb{R}$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "u^i"'] & C \arrow[d, "u^i"] \\ & \mathbb{R} \end{tikzcd}$$

$p \in \Delta(A)$ probability of the Action

$$\boxed{u_i(P) = \sum_{a \in A} p(a)\, u^i(g(a))}$$

$s \in S \equiv \bigtimes_{i=1}^{n} S^i = \bigtimes_{i=1}^{n} \Delta(A_i)$ where s is a mixed strategy profile

$s = (s^1, \dots, s^n)$ is a vector of mixed strategies.

$$\bigtimes_{i=1}^{n} \Delta(A_i) \xrightarrow{Pr} \Delta\left(\bigtimes_{i=1}^{n} A_i\right)$$

Probability of landing in an action profile.

$Pr_s \in \Delta(A)$ defined by

$$Pr_s(a) \equiv \prod_{i=1}^{n} s^i(a_i).$$

		q L	$1-q$ R
p	T	pq	$p(1-q)$
$1-p$	B	$(1-p)q$	$(1-p)(1-q)$

$$= Pr\big(\underbrace{(p, 1-p)}_{s^1}, \underbrace{(q, (1-q))}_{s^2}\big)$$

The function $s \longmapsto Pr_s$ is multilinear ; that is:

$\forall_{s^{-i} \in \times_{j \neq i} S^j}$ $\quad s^i \longmapsto Pr_{(s^i, s^{-i})}$ is linear.

<u>Proof:</u> take $(s^i, t^i) \in S^i$. Fix s^{-i}, $\theta \in [0,1]$.

$$Pr_{(\theta s^i + (1-\theta) t^i, s^{-i})}(a) = (\theta s^i + (1-\theta) t^i)(a^i) \prod_{j \neq i} s^j(a^j)$$

$$= \theta \, s^i(a^i) \prod_{j \neq i} s^j(a^j) + (1-\theta) t^i(a^i) \prod_{j \neq i} s^j(a^j).$$

$$= \theta \, Pr_{(s^i, s^{-i})} a + (1-\theta) Pr_{(t^i, s^{-i})} a$$

$\square$

Mixed Extension (ME)

<u>DEFINITION</u>: $(I, (S^i)_{i \in I}, (u^i)_{i \in I})$ is the ME of the NFG$_i$

, where: $\quad u^i(s) = \sum_{a \in A} Pr_s(a) \, u^i(a)$

		q	$1-q$
		L	R
p	T	3,1	0,0
$1-p$	B	0,0	1,3

$$u^1(\overbrace{\underbrace{p, 1-p}_{s_1}; \underbrace{q, 1-q}_{s_2}}^{s}) = 3pq + 1(1-p)(1-q)$$

$$u^i(s) = \sum_a \underbrace{\left[\prod_{i=1}^{n} s^i(a^i) \right]}_{Pr_s(a)} u^i(a)$$

Fact: For every $s^{-i} \in S^{-i}$, the function $s^i \longmapsto u^i(s^i, s^{-i})$ is linear.

Proof: Follows from $s^i \longmapsto Pr_{(s^j) s^i)}$ linear.

Best Response:

Take $i \in I$, $\bar{s}^{i} \in \bar{S}^{-i}$. solve:

$$BR_{A_i}(s^{-i}) = \max_{a^i \in A^i} u^i(a_i, s^{-i}) = \max_{a^i} \sum_{\bar{a}^{i} \in A^{-i}} \underbrace{P_{r_{s^{-i}}}(\bar{a}^{i})}_{\prod_{j \neq i} s_j(a_j)} u^i(a^i, \bar{a}^{i})$$

To get a <u>set</u> of best possible actions, call it $BR^i_{A^i}(s^{-i}) \left(\subseteq A^i \right)$

Properties of $BR^i_{A^i}$:

1. Since you are taking a maximum over a finite set:

$$\forall_{s^{-i}} \quad BR^i_{A^i}(s^{-i}) \neq \phi$$

2. If $\bar{s}^i(n) \longrightarrow s^{-i}$, $a^i \in BR^i_{A^i}\left(\bar{s}^i(n)\right)$ then $a_i \in BR^i_{A^i}(s^{-i})$.

<u>Proof:</u>

<u>Proof:</u> Since $a^i \in BR^i_{A_i}(\bar{s}^i_n)$:

$$u^i(a^i, \bar{s}^i_n) \geqslant u^i(b^i, \bar{s}^i_n) \quad \forall \quad b^i \in A^i.$$

since $u^i(a^i, x)$ is continuous in x:

$$\lim_{n \to \infty} u^i(a^i, \bar{s}^{i}_n) = u^i(a^i, s^{-i}) \geqslant u^i(b^i, s^{-i}) \quad \forall \quad b_i \in A_i. \qquad \square$$

Best Response:

For any $s^{-i} \in S^{-i}$. Let:

$$BR^i_{S^i}(s^{-i}) = \underset{S^i}{\arg\max}\ u^i(\cdot\,; s^{-i})$$

$S^i \equiv \Delta(A^i)$ = the set of optimal mixed strategies for i, given s^{-i}

$$= \{t^i \in S^i : u^i(t^i, s^{-i}) \geq u^i(v^i, s^{-i}) \ \forall_{v^i \in S^i}\}$$

this was already defined

Example:

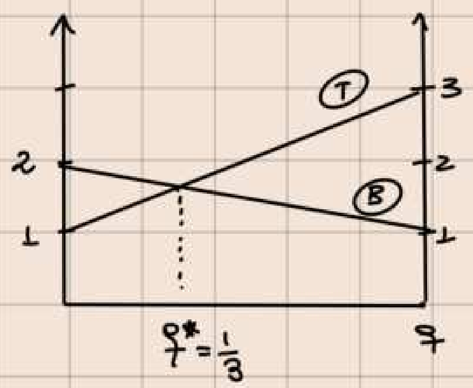
	q	$1-q$
	L	R
T	3	1
B	1	2

$s^{-i} = s^2 = (q, (1-q))$

$T \to 3q + 1 - q = 2q + 1$

$B \to q + 2(1-q) = 2 - q$

$$BR^1_{S^i}(q, 1-q) = \begin{cases} \{(0,1)\} & \text{if } q < 1/3 \\ \{(p, 1-p) : p \in [0,1]\} & \text{if } q = 1/3 \\ \{(1,0)\} & \text{if } q > 1/3 \end{cases}$$



Properties:

1. $\forall_{s^{-i}}\ BR^i_S(s^{-i}) = \phi$, closed and convex.

2. If $s^{-i}(n) \longrightarrow s^{-i}$, and $s^i \in BR^i_{S^i}(s^{-i}(n))$, then $s_i \in BR^i_{S^i}(s^{-i})$

Best Response

Link between $BR^i_{S^i}$ and $BR^i_{A^i}$:

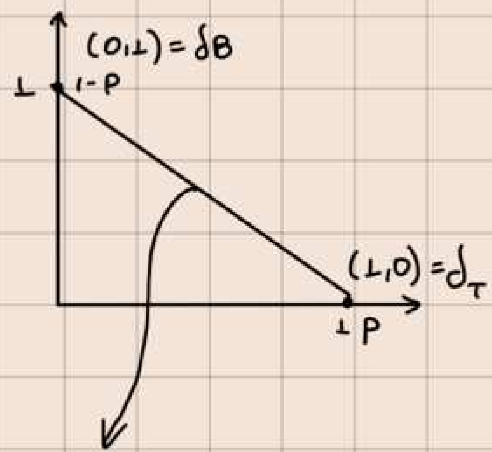
$$BR^1_{A^i}(q,(1-q)) = \begin{cases} \{B\} & \text{if } q < 1/3 \\ \{B,T\} & \text{if } q = 1/3 \\ \{T\} & \text{if } q < 1/3 . \end{cases}$$

What's the connection with?

Note $(0,1) = \delta_B$

$$B_R{}^1_{S^i}(q,1-q) = \begin{cases} \{(0,1)\} & \text{if } q < 1/3 \\ \{(p,1-p) : p \in [0,1]\} & \text{if } q = 1/3 . \\ \{(1,0)\} & \text{if } q > 1/3 \end{cases}$$

δ_T



$$S^1 = \{ \theta \delta_T + (1-\theta)\delta_B : \theta \in [0,1] \}.$$
$$= \text{convex hull of } \{\delta_B, \delta_T\}$$

Definition: X is a finite set, eg $X = \{s^1, t^1, v^1\}$

$$co(X) = \{ s = \sum_{j=1}^{K} \theta_j x_j \quad \forall i \ \theta_j \geq 0, \ \sum_j \theta_j = 1 \}$$

Fact:

$$BR^i_{S^i}(s^{-i}) = co\left(\{ \delta_{a^i} : a^i \in BR^i_{A^i}(s^{-i}) \} \right)$$

③ $BR^i_{S^i}(s^{-i}) = co\left(\{ \delta_a^i : a^i \in BR^i_{A^i}(s^{-i}) \}\right)$.

Denote $$u^i(s) = \sum_{a \in A} Pr_s(a)\, u^i(a)$$

$$= \sum_{a^i \in A^i} s^i(a^i) \sum_{\bar{a}^i \in A^{-i}} Pr_{\bar{s}^i}(\bar{a}^{-i})\, u^i(a^i; s^{-i})$$

$$= \sum_{a^i \in A^i} s^i(a^i)\, u^i(a^i, s^{-i}) = u^i(s^i, s^{-i}).$$

Define: $$BR^i_{A^i}(s^{-i}) = \arg\max_{a^i \in A^i} \{ u^i(a^i, s^{-i}) \}$$

$$\equiv \left\{ a^i \in A^i \,\middle|\, \forall b^i \in A^i,\ u^i(a^i, s^{-i}) \geq u^i(b^i, s^{-i}) \right\}$$

Define: $$BR^i_{S^i}(s^{-i}) = \arg\max_{s^i \in S^i} \sum_{a^i \in A^i} s^i(a^i)\, u^i(a^i, s^{-i})$$

$$= \left\{ s^i \in S^i \,\middle|\, t^i \in S^i;\ u(s^i, s^{-i}) \geq u(t^i, s^{-i}) \right\}$$

(⇐)

Take s^{-i} as given. Take $a^i \in BR^i_{A^i}(s^{-i})$, then by definition:

$$\forall c^i \in A^i : \quad u^i(a^i, s^{-i}) \geq u(c^i; s^{-i})$$

$$\forall s^i \in S^i : \quad u^i(a^i; s^{-i}) \geq \sum_{c^i \in A^i} s^i(c^i)\, u(c^i, s^{-i})$$

Note that we can rewrite last inequality as:

$$\forall s^i \in S^i \quad u^i(\delta_a^i, s^{-i}) \geq \sum_{c^i \in A} s^i(c^i)\, u(c^i, s^{-i})$$

Therefore, $\delta_a^i \in BR^i_{S^i}(s^{-i})$ for all $a^i \in BR^i_{A^i}(s^{-i})$.

Now, Assuming that A^i is finite; $BR^i_{A^i}(s^{-i}) \neq \phi$ and finite.

Take $\Delta^* = \Delta^{\# BR^i_{A_i}(s^{-i})}$. Note that $\#\Delta^* \leq m$. For $\tau \in \Delta^*$;

$\forall\ s^i \in S^i$: $$\sum \tau_i\ \mu^i(d_{a^i}, s^{-i}) \geq \sum_{a^i \in A_i} s_i(a^i)\ \mu^i(a^i, s^{-i})$$

Since every $\sum \tau_i d_{a^i} \in S$, It also belongs to $BR^i_{S^i}(s^{-i})$

$\Rightarrow)$ $BR_i(s) \subset co\left(\{ d_{b^i} : b^i \in BR^i_{A^i}(s) \}\right)$.

Lemma 1: $\forall\ b^i \notin BR_{A_i}(s^{-i})$, $b^i \in A_i \Rightarrow s^i(b^i) = 0\ \forall\ BR_{S^i}(s^{-i})$.

Proof: Suppose not. Take $s^i \in BR_{S^i}(s^{-i})$, with $s^i(b^i) > 0$. Since $b^i \in BR_{A_i}(s^{-i})$ $\exists\ c^i \in A^i$ such that $\mu^i(c^i, s^{-i}) > \mu^i(b^i, s^{-i})$. Consider $s^{i\prime}$ where we $s^{i\prime}(c^i) = s^i(c^i) + s^i(b^i)$ and $s^{i\prime}(b^i) = 0$. Then:

$$\sum s^{i\prime}(a^i)\ \mu(a^i, s^{-i}) = \sum_{a^i \neq b^i, c^i} s^i(a^i)\ \mu(a^i, s^{-i}) + s^{i\prime}(c^i)\ \mu(c^i, s^{-i}) + s^{i\prime}(b^i)\ \mu(b^i, s^{-i})$$

$$> \sum s^i(a^i)\ \mu(a^i, s^{-i}). \ \lightning$$

Therefore $\forall\ s^i \in BR_{S^i}(s^{-i})$ $s^i(a^i) > 0$ only if $a^i \in BR_A(s^{-i})$. Since $\sum s^i(a^i) = 1$ we know that $s^i \in co\{(d_{a^i} : a^i \in BR_{A_i}(s^{-i})\}$.

27-01

Lecture 4

Best Response:

Theorem:

$$\forall_{i, s^{-i}} \quad BR^i_{S^i}(s^{-i}) = co\left(\{ \delta_{a^i} : a^i \in BR^i_{A^i}(s^{-i}) \} \right)$$

$$co(X) = \bigcap_{C \supseteq X} \{ C \mid C \text{ convex} \}$$

$\boxed{\text{Nash Equilibria}}$

$$BR^i_{S^i} : S^{-i} \rightrightarrows S^i$$

$$s^{-i} \longrightarrow BR^i_{S^{-i}}(s^{-i})$$

non-empty, closed, convex and:

1. $s^{-i}(n) \longrightarrow s^{-i}$
2. $s^i(n) \in BR^i_{S^i}(s^{-i}(n)) \ \forall n$
3. $s^i(n) \longrightarrow s^i$

closed graph - uhc.

Then $s^i \in BR^i_{S^{-i}}(s^{-i})$

Definition: A Nash Equilibrium is an $s^* \in S$ st:

$$\forall i \quad s_i^* \in BR^i_{s^{-i}}(s^{*-i})$$

Theorem: The set of the NE of a finite NFG is non-empty

finite: $\# I < +\infty$

$\forall i \quad \# A_i < +\infty$

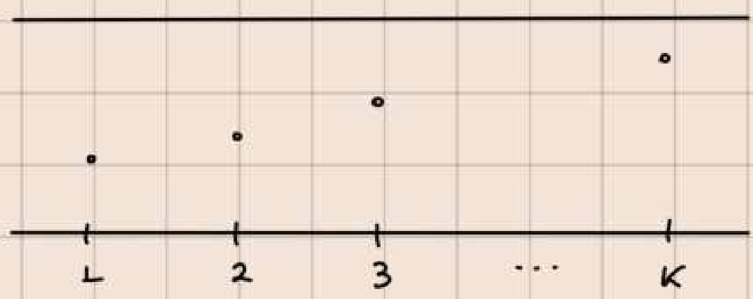
what does finite mean?

Old Prelim Question

Example: $\# J = 1 \quad A^1 = \{1, 2, 3, \dots, K, \dots\}$

$NE = \emptyset$ if $u^i(k) = 1 - \frac{1}{k}$.

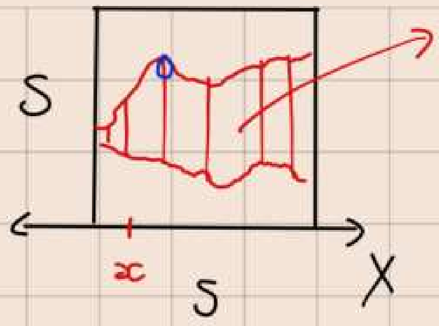
There is no dominant strategy to follow, nor nash equilibrium, even in mixed strategies.



Kakutani's Fixed Point theorem

Definition: $\Phi : S \rightrightarrows S$ has a closed graph if:

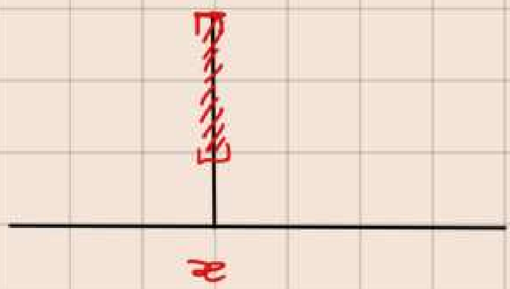
$$\Gamma_\Phi \equiv \bigcup_{x \in S} (x, \Phi(x)) \text{ is a closed set.}$$



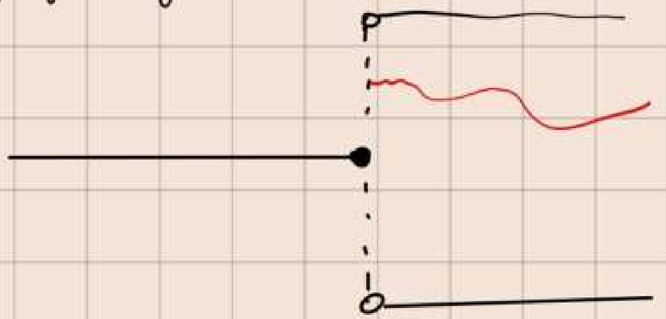
Γ_Φ the graph of Φ

Assume that point is not there.

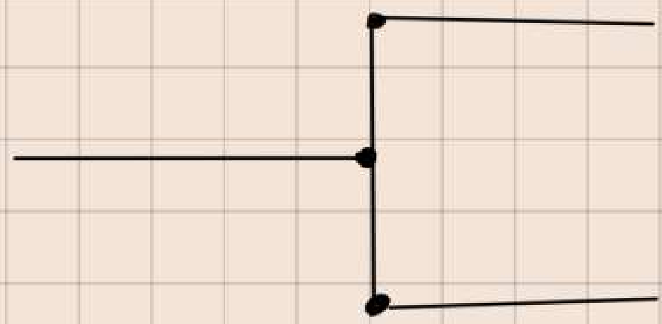
For the correspondence to have a closed graph you need that for any x:



or it may fail if:



There is a sequence that converges to a point outside the correspondence



Fact: Γ_Φ is closed iff

$\forall_{x_\delta} \longrightarrow x$, $y_\delta \in \Phi(x_\delta)$ where $y_\delta \to y$,

$\Longrightarrow y \in \Phi(x)$.

Theorem: Let $X = \mathbb{R}^n$ ⊛, S a compact, convex subset of X. $\Phi : S \twoheadrightarrow S$ such that Γ_Φ is closed, $\forall_x\ \Phi(x) \neq \emptyset$, convex, then $\exists_{s^* \in S}\ s^* \in \Phi(s^*)$.

(convex valued means that $\forall x$, $\Phi(x)$ is a convex set)

Proof of the theorem:

$$\overline{BR}^i_{s^i}(s) = BR^i_{s^i}(s^{-i})$$

$$s^{-i} = (s^2, \dots, s^n)$$

Think of it as having a suggestion on what to do, that you ignore. It's a trick to make things easier.

$$\overline{BR}^1_{s^1}(s_1, s_2, \dots, s_n) = BR^1_{s^1}(s_2, \dots, s_n)$$

$$\Phi = BR_s(s) = \begin{bmatrix} \overline{BR}^1_{s^1}(s) \\ \overline{BR}^2_{s_2}(s) \\ \vdots \\ BR^n_{s_n}(s) \end{bmatrix}$$

$$s \in S \equiv \bigtimes_{i=1}^{n} s^i$$

$$\Phi(s) = \bigtimes_{i=1}^{n} \overline{BR}^i_{s^i}(s^{-i})$$

Prove that this is convex, possible prelim question.

one can prove:

1. $\bigtimes_{i=1}^{n} S^i \equiv S$ is compact convex subset of $\mathbb{R}^m$ ($m = ?$)

2. $\forall i$ $\overline{BR}^{\,i}_{s^i}$ satisfies properties of $BR^i_{s^i}$

3. The cartesian product of correspondences that satisfy the properties (convex valued, non empty, closed graph). also satisfies those properties.

$\Rightarrow$ then we can use KFT, $\exists_{s^* \in S}$ $s^* \in BR_S(s^*)$

$\square$

Exercise 3

Problem 3. *Let BR_{S^i} be the best response in mixed strategies.*

1. *Prove that the correspondence*
$$s^{-i} \mapsto BR_{S^i}(s^{-i})$$
is closed graph.

2. *Prove in detail all the steps needed to show that the set S and the correspondence BR_S satisfy the conditions needed to apply the theorem on existence of a fixed point.*

$BR : S^{-i} \rightrightarrows S^i$

1 wts that $s^{-i} \mapsto BR_{S^i}(s^{-i})$ has a closed graph.

w.t.s $\bigcup_{s^{-i} \in S^{-i}} (s^{-i}, BR(s^{-i}))$ is a closed set.

The graph of the correspondence:

$$Gr(BR_{S^i}(s^{-i})) = \{ (s^{-i}, s^i) : s^{-i} \in S^{-i}, s^i \in BR_{S^i}(s^{-i}) \}$$

$Gr(BR_{S^i}(s^{-i}))$ has a closed graph iff:

$$\forall s^{-i}_{(n)} \to s^{-i}, \ \forall s^i(n) \to s^{i*} : s^i(n) \in BR_{S^i}(s^{-i}(n)) \Rightarrow s^i \in BR_{S^i}(\bar{s}^{i}).$$

Take $\{s_n\}$ st. $\{s_n^{-i}\} \to \{s^{-i}\}$, (*) we also know that the convergence point wise: $s_n^i(a_i) \to s_i^*(a_i)$. Then

$$\forall \tilde{s}^i \in S_i : \sum s_n^i(a^i)\, u(a^i; s^{-i}(n)) \geq \sum \tilde{s}^i(a_i)\, u(a_i, s^{-i}(n))$$

Since u is continuos:

$$\forall \tilde{s}_i \in S_i : \sum s^{i*}(a_i)\, u(a^i; s^{-i}) \geq \sum \tilde{s}_i(a_i)\, u(a_i, s^{-i}).$$

Hence $s^{i*} \in Br(s^{-i})$. □

(*) working with the metric $d(x_1, x_2) = \sqrt{(x_{11} - x_{12})^2 + (x_{21} - x_{22})^2 + \dots + (x_{n1} - x_{n2})^2}$

② Recall that the best response correspondence was defined as follows:

$$BR : S \rightrightarrows S \quad \text{where} \quad S = \bigtimes_{i=1}^{N} S_i$$

$$BR(s) = \left(BR^1_{S_1}(s) \, ; \, BR^2_{S_2}(s) \, ; \dots ; \, BR^N_{S_I}(s) \right)$$

where $BR^i_{S_i} : S \to S$ is such that, iff $s \in BR^i_{S_i}$:

$$\forall \; t_i \in S_i \, ; \quad \underbrace{\sum_{a_i \in A_i} s^i(a_i) \, u(a_i, s^{-i})}_{u(s^i, s^{-i})} \geq \underbrace{\sum_{a_i \in A_i} t_i(a_i) \, u(a_i ; s^{-i})}_{u(t_i, s^{-i})} .$$

We want to apply the following theorem:

Let $\Gamma : X \rightrightarrows X$ be a correspondence. Let: • X be - nonempty - compact - convex   • Γ be: - nonempty - convex valued - u.h.c $\Big\} \Rightarrow$ closed graph $\underbrace{X \text{ compact} \quad \Gamma \text{ uhc}}_{\Uparrow}$ $\quad$ $\underbrace{\Gamma \text{ closed} \quad \Gamma \text{ uhc}}_{\Uparrow}$ Then $\exists \; x^* \in X$, st $x^* \in \Gamma(x^*)$.

We will show that $BR : S \rightrightarrows S$ and S satisfy the previous conditions.

Note that $\forall i$, S^i is a simplex. Therefore is a nonempty compact and convex set. By the following proposition the cartesian product of non-empty compact and convexs sets is non-empty compact and convex we are done.

Proposition: the cartesian product of $\boxed{S_1 \text{ and } S_2 \text{ is compact and convex}}$.

KKT

Proof: A set is compact if from any sequence of elements S you can extract a sub-sequence with a limit in S. Therefore take a sequence (S_n) in $S_1 \times S_2$. We can write this as $(s_1(n), s_2(n))$. Since $\{s_1(n)\}$ is a sequence in S_1 we can find $\{s_1(n_k)\} \to s_1 \in S_1$. By the same Argument we can find a convergent subsequence in S_2. Thus now construct the subsequence $\{s_1(n_k), s_2(n_k)\} = \{s_{n_k}\} \to s \in S_1 \times S_2$.

Now, take $s_1, \tilde{s}_1 \in S_1$ and $s_2, \tilde{s}_2 \in S_2$. where $(s_1, s_2), (\tilde{s}_1, \tilde{s}_2) \in S_1 \times S_2$. take $(\alpha s_1 + (1-\alpha)\tilde{s}_1, \alpha s_2 + (1-\alpha)\tilde{s}_2) \in S_1 \times S_2$ since S_1 and S_2 are simplex.

Note that the proof for $\bigtimes_{i=1}^{I} S_i$ is a direct extension.

Now we will check the conditions over the correspondence:

$\boxed{\circ\ BR(s) \text{ is Non-empty}}$ KKT

$\forall i$, $\max_{s_i \in S_i} u(s_i, s^{-i})$ is well defined since we are maximizing a continuous function over a compact set. Hence $\forall i, \exists\, s_i \in BR_{S_i}(s)$.

$\boxed{\circ\ BR(S) \text{ has a closed graph.}}$

∘ $BR(s)$ is Compact valued and convex:

∘ $BR^i_{S_i}(s^{-i})$ is closed: Fix $s^{-i} \in S^{-i}$. Then construct $\{s^i(n)\} \in BR^i_{S_i}(s^{-i})$ st $\{s^i(n)\} \to s^i$. By definition $\forall\, t^i \in S^i$, $u(s^i(n); s^{-i}) \geq u(t^i, s^{-i})$. Since u is continuous we know $u(s^i; s^{-i}) \geq u(t^i, s^{-i})\ \forall\, t^i \in S^i$. □

∘ $BR^i_{S_i}(s^{-i}) \subset S_i$ which is Bounded by definition □

By the last proposition. we know that the cartesian product of compact sets is compact. Hence $\forall\, s^{-i}$ $BR(s^{-i})$ is a compact set.

• $BR(s^{-i})$ is convex. We know that the cartesian product of convex sets is convex. Then If we show that $BR_{S_i}(s^{-i})$ is convex we are done. Take $s_i, \tilde{s_i} \in BR^i(s^{-i})$. Then by definition: $\mu(s_i, s^{-i}) \geq \mu(t_i, s^{-i})$ and $\mu(\tilde{s_i}, s^{-i}) \geq \mu(t_i, s^{-i})$ $\forall t_i$. Take $\alpha s_i + (1-\alpha) \tilde{s_i}$ and the result will still hold $\forall$ $t_i \in S_i$. $\square$

• $BR(s)$ is uhc:

• $BR^i(s^{-i})$ is uhc: since we have shown that $BR^i(s)$ is compact valued we only need to show that:

$\forall \{s^{-i}(n)\} \to s^{-i}$, $\{s^i(n)\}$ where $s^i(n) \in BR^i_{S_i}(s^{-i}(n))$ $\exists$ a convergent subsequence $\{s^i(n)_k\} \to s_i$ where $s^i \in BR_{S_i}(s^{-i})$. First, we know that since S_i is compact $\exists$ $\{s^i(n)_k\} \to s_i$ where $s_i \in S_i$. Now we also know that $\mu(s^i(n)_k, s^{-i}(n_k)) \geq \mu(t_i, s^{-i}(n_k))$ $\forall$ $t_i \in S_i$. Since μ is continuous $\mu(s^i; s^{-i}) \geq \mu(t_i, s^{-i})$ $\forall$ $t_i \in S_i$ $\square$.

• Since $BR(s)$ is compact-valued we want to show $\forall$ $\{s(n)\} \to s$, $\{s^*(n)\}$ where $\{s^*(n)\} \in BR(s(n))$ $\exists$ a convergent subsequence $\{s^*(n)_k\} \to s^*$ where $s^* \in BR(s)$. Since S is compact we know s^* and $\{s^*(n)_k\}$ $\exists$. wts that $s^* \in BR(s)$. Note that $s^* = (s_1^*, \dots, s_I^*)$ and $\{s^*(n)_k\} \to s^* \Rightarrow \{s^*(n)^i_k\} \to s^{*i}$ $\forall$ $i = 1, \dots, I$ (under some normal distance) Since $s^{*i}_{nk} \in BR^i(s_{nk})$ we have showed that $s^{*i} \in BR^i(s)$ $\forall i$. Hence $s^* \in Br(s)$.

closed and uhc $\Rightarrow$ closed graph
But If we add compact we can have an easier definition for uhc.

Refinements of NE:

Perfect Equilibria:

	L	R
T	1,1	0,0
B	0,0	0,0

In this game, (B,R) in pure strategies is a NE. NOT PLAUSIBLE.

Idea of Refinements: A NE is plausible if it is robust to small perturbations.

I. Perturbations of utilities:

II Perturbations of strategies:

Perturbations of utilities:

		q	$1-q$
		L	R
p	T	1,1	0,0
$1-p$	B	0,0	x,y

Ⓛ $\to p$

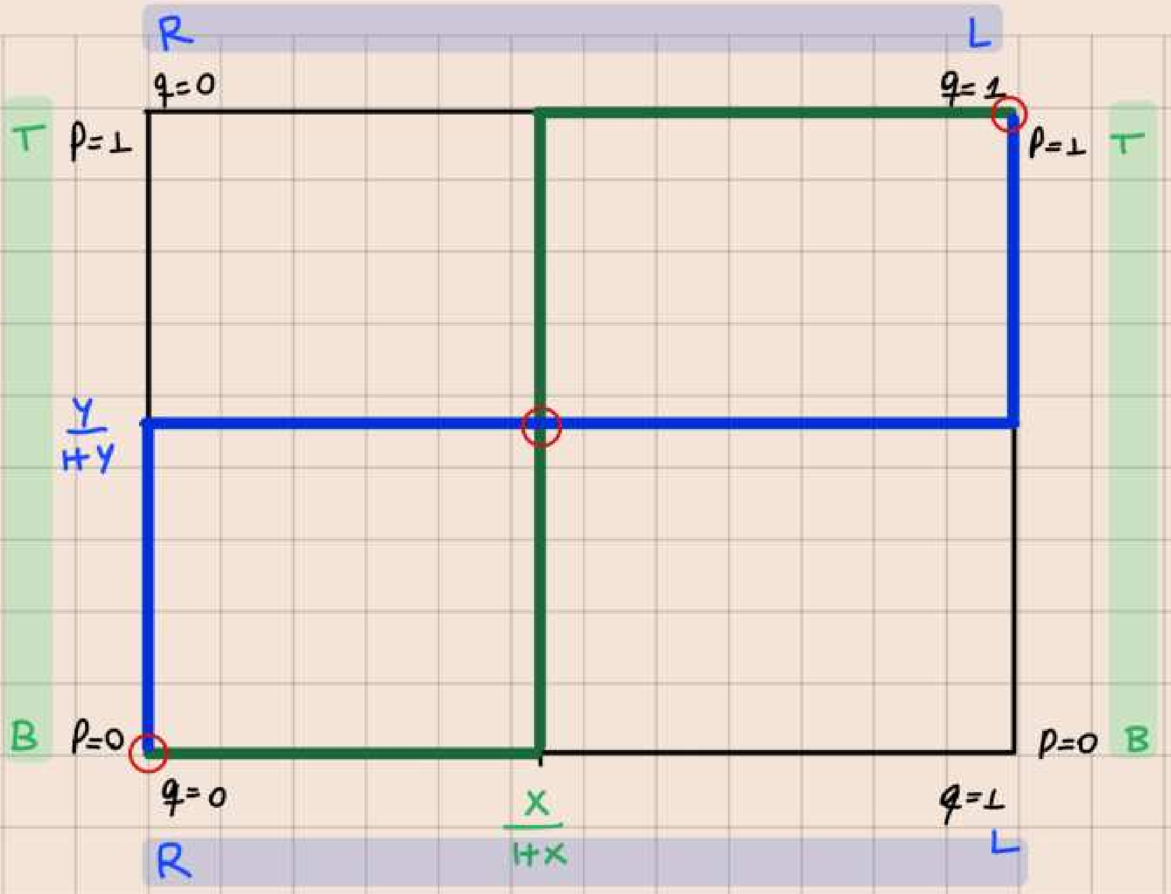
Ⓡ $\to (1-p)\, y$.

What happens if we change $(x,y) \in \mathbb{R}^2$ a little bit?

what is $BR^1_{S^1}((q; 1-q)) \equiv BR^1_{S^1}(q)$

Ⓣ $\to q$ Ⓑ $\to (1-q)\, x$.

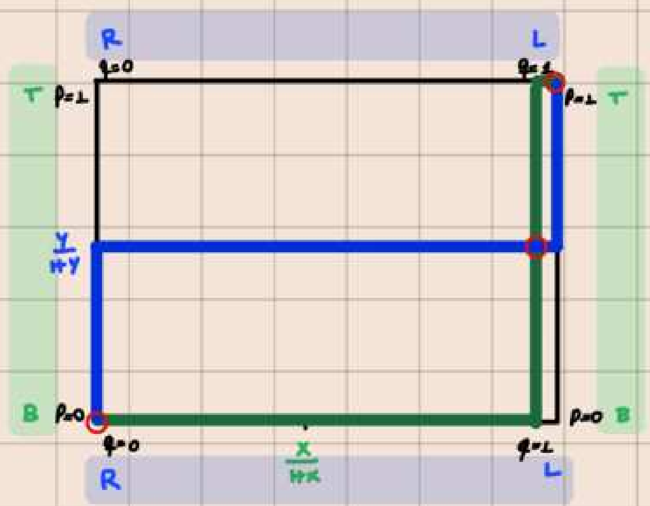
x	$BR^1_{S^1}(q)$
$x<0$	$\{1\}$ Agent always want to play T.
$x=0$	$q>0$ $\{1\}$; $q=0$ $[0,1]$
$x>0$	$q^* = \frac{x}{1-x}$ then if $q>q^*$ $\{1\}$; $q=q^*$ $[0,1]$; $q<q^*$ $\{0\}$.



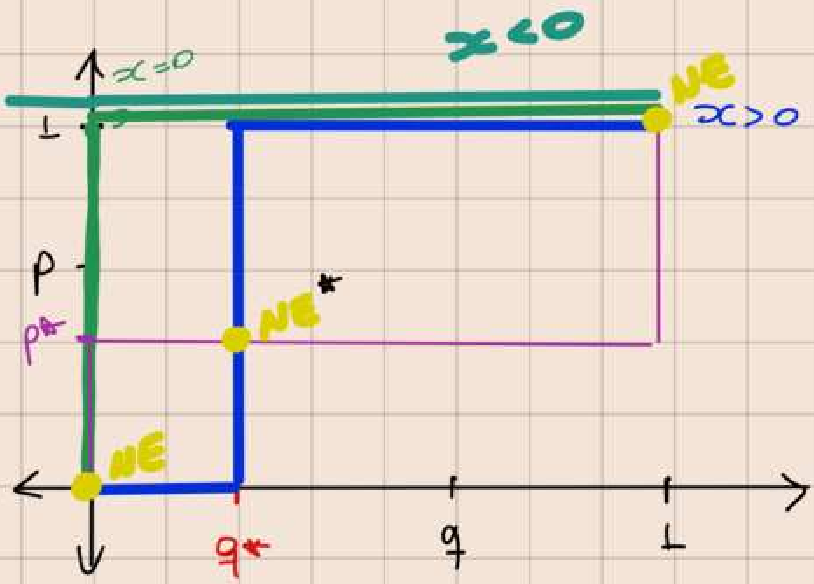
Remark: we can move x to the left and to the right.

If we move it to the left enough we will make this game have only one equilibrium. For values of x or y low enough, agents start to have a strictly dominant strategy.

If we move it to the right we are always going to have the same amount of equilibrios.

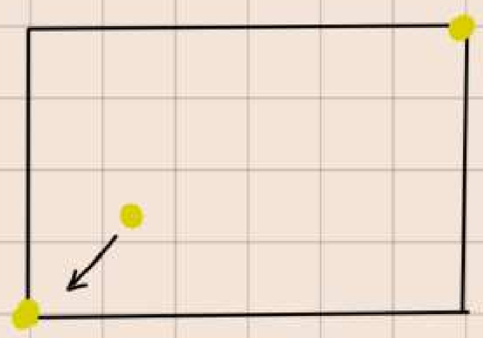


Graph of the Best Response:



For the other agent $p^* = \frac{y}{1+y}$

If y becomes smaller the NE* goes down until it goes away.



you will have both in the same corner once $y=0$

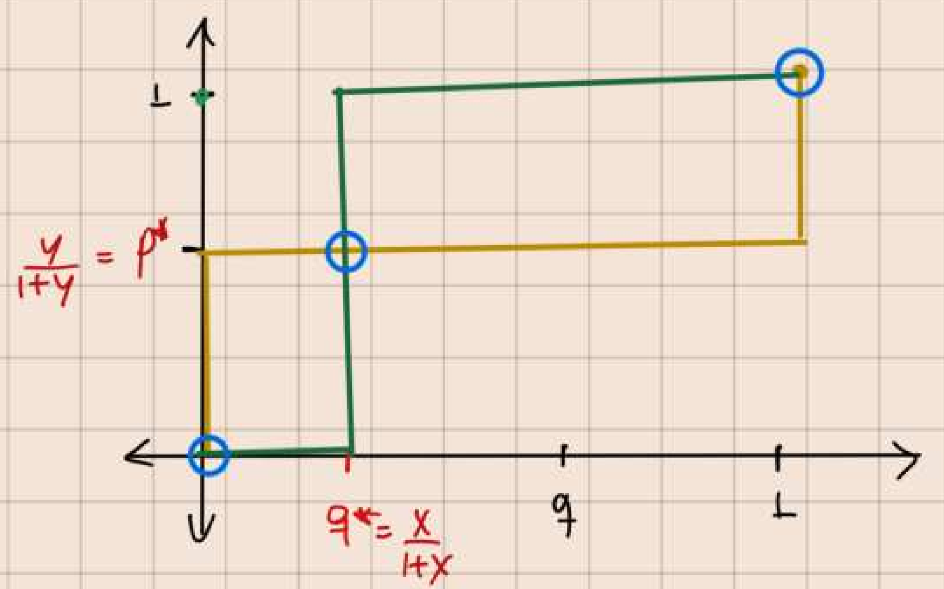
If $(x,y)<0$ the only Nash Equilibria that remains is:

this is the only plausible Nash Equilibria, It still exists still if you take perturbations, no matter how much you shake.

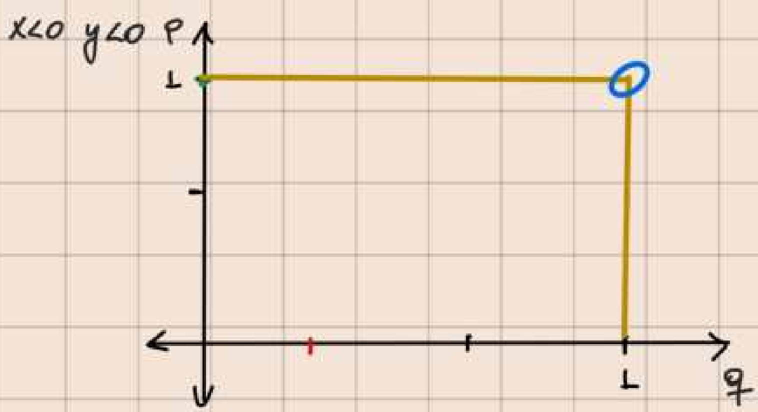
$x>0 \quad y>0$

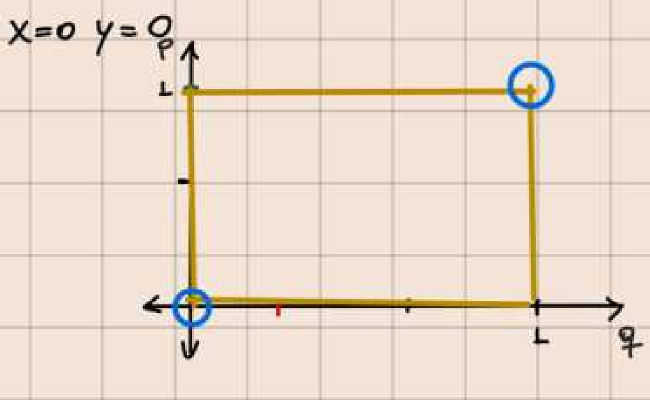
L $\rightarrow p$

R $\rightarrow (1-p)y$



y	$BR_{s^2}^2(p) = q$
$y<0$	$\{1\}$ Agent always want to play T.
$y=0$	$p>0 \quad \{1\}$ $p=0 \quad [0,1]$
$y>0$	$p^* = \frac{y}{1+y}$ then if $p>p^* \quad \{1\}$ $p=p^* \quad [0,1]$ $p<p^* \quad \{0\}$.





In this Example perturbations of utilities select (T, L) as the unique perfect equilibrium.

II. PERTURBATIONS OF STRATEGIES:

Idea: player cannot choose an action <u>for sure</u>

If you choose:

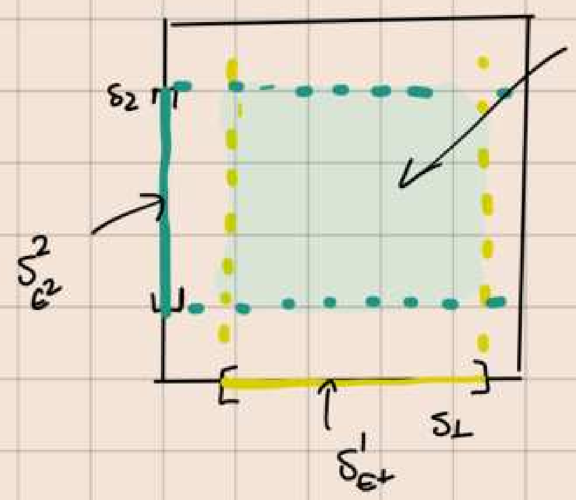
But you cannot choose this, you can only choose an element in the green set:

$$\left\{p: \ \varepsilon^1 \leq p \leq 1-\varepsilon^2\right\}$$

Definition for $\varepsilon^i > 0$ small $\left(\# A_i \cdot \varepsilon \leq 1 \right)$

$$S^i_{\varepsilon^i} = \left\{ s^i \in S^i : \ \forall_{a^i \in A^i} \quad 1-\varepsilon^i \geq s^i(a^i) \geq \varepsilon^i \right\}$$

$$S_\varepsilon = \bigtimes_{i=1}^{n} S^i_{\varepsilon^i}$$



S_ε the mix strategies are all going to be in this set.

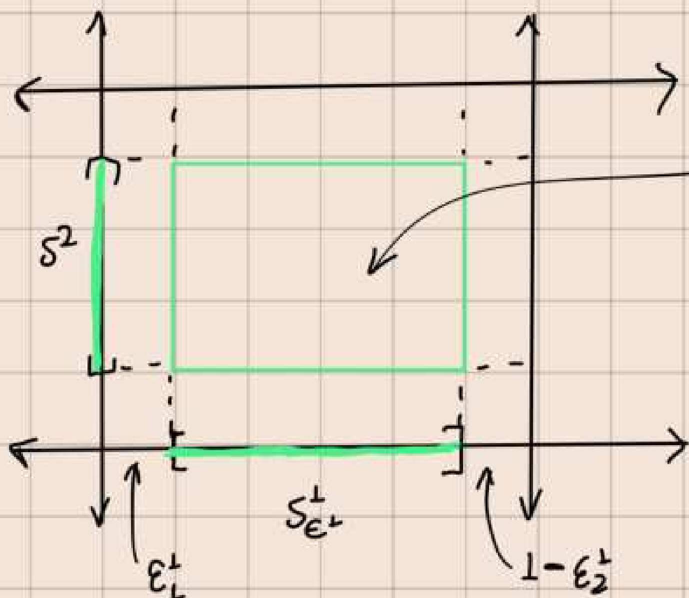
Using a methodical or a clever way we can prove that the NE of the perturbated game exists.

DEFINITION: $S^i_{\epsilon i} = \{ s^i \in S^i : \forall_i \ s^i(a^i) \geq \varepsilon^i_i \}$ for $\varepsilon^i = (\varepsilon^i_1, \dots, \varepsilon^i_{\#A_i})$ st $\forall_i \ \varepsilon^i_i > 0 \ \sum_i \varepsilon^i_i < 1$ is defined as the set of perturbed strategies. (*)

DEFINITION: the perturbed game for $\varepsilon = (\varepsilon^i)_{i \in I}$:

$$\left(I, \left(S^i_{\epsilon i}\right)_{i \in I}, u^i \Big| \bigtimes_{i=1}^{n} S^i_{\epsilon i} \right) \text{ where } \varepsilon^i = (\varepsilon^i_1, \dots, \varepsilon^i_{\#A_i})$$

satisfies (*) for all i.



$$u^i \Big|_{\bigtimes_{i=1}^{n} S^i_{\epsilon i}} : \bigtimes_{i=1}^{n} S^i_{\epsilon i} \longrightarrow \mathbb{R}.$$

Theorem: For all ϵ there is a Nash Equilibrium of the perturbed game.

Proof: Note that the set S_ε of perturbed strategies is non-empty compact valued and convex, hence the Kakutani Fixed Point theorem still applies.

DEFINITION: constrained BR is:

$$BR^i_{S^i_{\varepsilon_i}}(s^{-i}) = \underset{s^i \in S^i_{\varepsilon^i}}{\arg\max}\ u^i(s^i, s^{-i})$$

DEFINITION: A perfect Equilibrium of a NFG is an $s^* \in S$ such that:

$\exists_{\varepsilon(n)}$ satisfying $(*)$ ($\varepsilon(n)$ sequence of perturbations)

such that

$$\varepsilon(n) \longrightarrow 0 \in \mathbb{R}^{\#A_1 \times \#A_2 \dots \times \#A_n}$$

and $s^*_{\varepsilon(n)}$ equilibrium of the perturbed game, st:

$$s^*_{\varepsilon(n)} \longrightarrow s^*.$$

The only perfect Equilibrium is (T,L). Nothing else can be we can check the set of $NE \subset (T,L)^c$.

Fact: A perfect Equilibrium is a Nash Equilibrium.

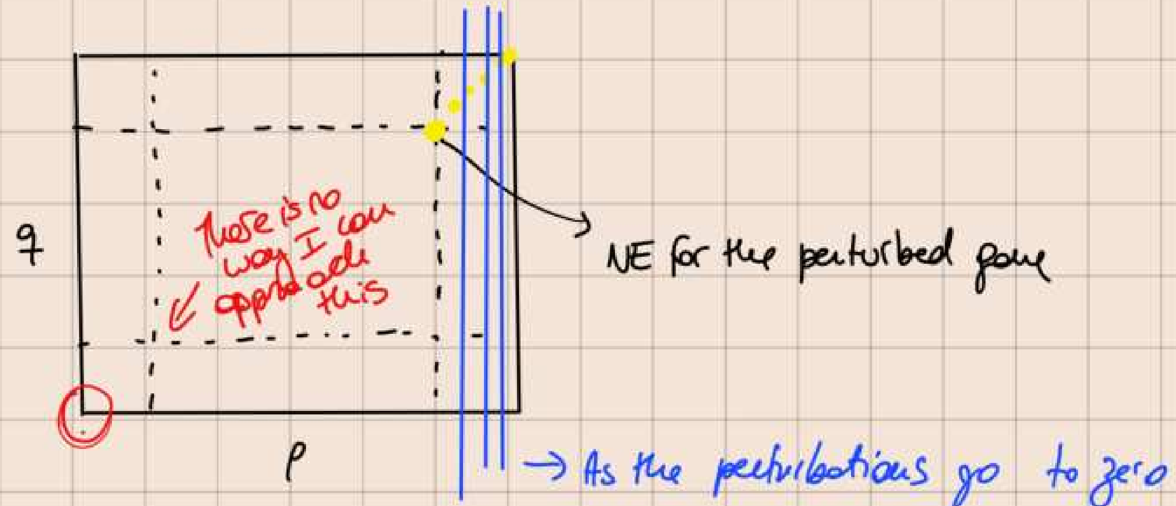
	L	R
T	1,1	0,0
B	0,0	0,0

Since $PE \subset NE$ then we only need to analyze the other NE

Proving that (T, L) is the only PE:

Step 1: (T, L) is PE

Step 2: Use Fact 2 and Prove (B, R) is not PE.



	ε_2 L	$1-\varepsilon_2$ R
T	1,1	0,0
B	0,0	0,0

$T \rightarrow q \cdot 1 + (1-q)0 \geq \varepsilon_2 1$

$B \rightarrow 0$

He cannot choose T with probability 1.

$$BR^{1}_{S^1_{\varepsilon^1}}(\bar{s}^{2}) = \left\{ (1-\varepsilon_1, \varepsilon_1) \right\} \text{ for } \bar{s}^2 \in S^2_{\varepsilon^2}.$$

THEOREM: The set of Perfect Equilibria of a finite NFG is a non-empty subset of the set of Nash Equilibria. ($PE \subset NE$)

Proof: For any sequence: $\varepsilon(n) \longrightarrow 0$, there is a non-empty set of NE for the perturbed game $NE(\varepsilon(n))$ (since KT fixed point theorem follows). Pick for every n:

$$s^*(n) \in N(\epsilon(n))$$

$\{s^*(n)\}$ is a sequence in the compact set S, which has an accumulation point s^*.

$PE \subset NE$

Let $s^* \in S$ be a PE. $\exists$ $\{\epsilon_n, s_n^*\}$ such that $\epsilon_n \longrightarrow 0$
$s_n^* \longrightarrow s^*$ and $s_n^* \in NE(\Gamma_{\epsilon_n})$ $\forall$ n.

$\forall$ i:

$$u^i(s_n^{i*}, s_n^{-i*}) \geq u^i(t_n^i; s_n^{-i*}) \quad \forall \quad t_n^i \in S_{\epsilon_n}^i.$$

Wts:

$$u^i(s^{i*}, s^{-i*}) \geq u^i(t^i, s^{-i*}) \quad \forall \quad t_i \in S_i.$$

for any t_i, pick $t_n^i \longrightarrow t^i$. We then know that

$$u^i(s_n^{i*}, s_n^{-i*}) \geq u_i(t_n^i, s_n^{-i*})$$

$$u^i(s^{i*}, s^{-i*}) \geq u_i(t^i, s^{-i*})$$

therefore since t was arbitrary, s^* is a $NE(\Gamma)$.

$NE \not\subset NE$. Look at the previous example.

$\square$

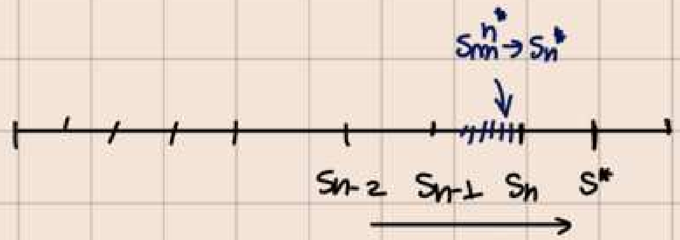
Proposition: $PE(\Gamma)$ is closed.

Proof: Take any sequence of $\{s_n^*\}$ where s_n^* is a PE st $s_n^* \to s^*$.

w.t.s s^* is a PE. $\exists \{\varepsilon_N\}, \{s_N^*\}$ where $s_N^* \in \Gamma(\varepsilon_N)$ and $\varepsilon_N \to 0$ and $s_N \to s^*$.

for each n, $\exists$ $s_m^{n*} \to s_n^*$ where $s_m^{n*} \in NE(\varepsilon_m)$ where $\varepsilon_m \to 0$.

Idea:



Since $\{s_n^*\}$ are all NE by continuity s^* is a NE.

We only need to construct the sequence of perturbed strategies.

Construct the sequence of strategies in the following way:

(For simplicity assume perturbations do not depend on the strat chosen).

1. start from s_1^*. take the first perturbed eq of the sequence of NE of the perturbed games.
2. Now, Take s_m^* at least at half the distance between s_1^* and s^*. since s_m^* is a PE, $\exists$ a perturbed strategy for a perturbation smaller than in step 1.
3. Repeat and generate $\{\varepsilon_m\} \to 0$ $\{s_m^*\} \to s^*$.

Bipul's recitation

Claim: In any PE $s^*(a^i) = 0$ for any a^i weakly dominated.

Proof: Let s^* be a PE.

$\exists \ \{\varepsilon_n, s_n^*\}$ st $s_n^* \in NE(\Gamma_{\varepsilon_n})$ with $s_n^* \to s^*$ $\varepsilon_n \to 0$.

Since a_i is weakly dominated $s_n^{i*}(a_i) = \varepsilon_n$.

As $\varepsilon_n \to 0$ $s_n^{i*}(a^i) \longrightarrow 0$, therefore $s_i^*(a^i) = 0$.

Lesson: We can dismiss NE that play weakly dominated strategies when we are computing PE.

Theorem: A fully mixed NE profile is a PE.

Let s^* be a NE such that $\forall i, \forall a^i \in A^i$:

$s_i^*(a_i) > 0$.

Define: $\underline{s}^i = \min\limits_{a_i} s_i^*(a_i)$ $\quad \underline{s} = \min\limits_{i \in I} \underline{s}^i$

Choose $\varepsilon_n = \frac{\varepsilon}{n}$ where $0 < \varepsilon < \underline{s}$.

Since $s_n^* \in NE(\Gamma_{\varepsilon_n})$ $\forall n$, $s_n^* \to s^*$.

$\square$

Correlated equilibra

Recall $s = (s^1, \dots, s^n) \in \bigtimes_{i=1}^{n} \Delta(A^i) \xrightarrow{Pr} \Delta\left(\bigtimes_{i=1}^{n} A_i\right)$

where $Pr : S \longrightarrow Pr(a)$.

Example: Battle of the sexes.

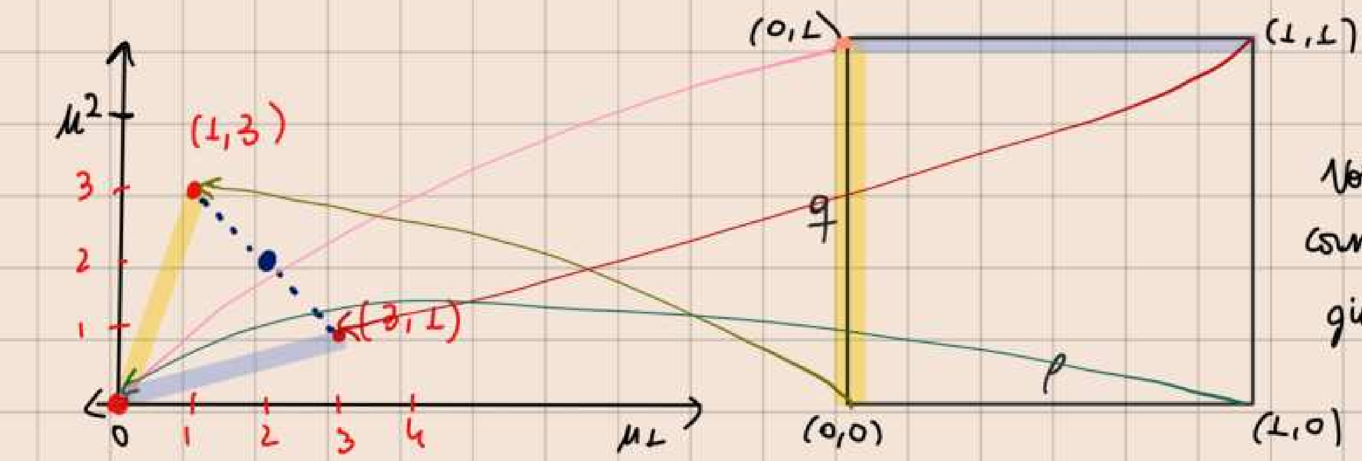
		μ_2	
		q	$(1-q)$
		L	R
μ_1 p	T	3,1	0,0
$(1-p)$	B	0,0	1,3

$(p.3 \,;\, p.1) = p(3,1)$

$((1-q).1 \,;\, (1-q)3) = (1-q)(1,3)$

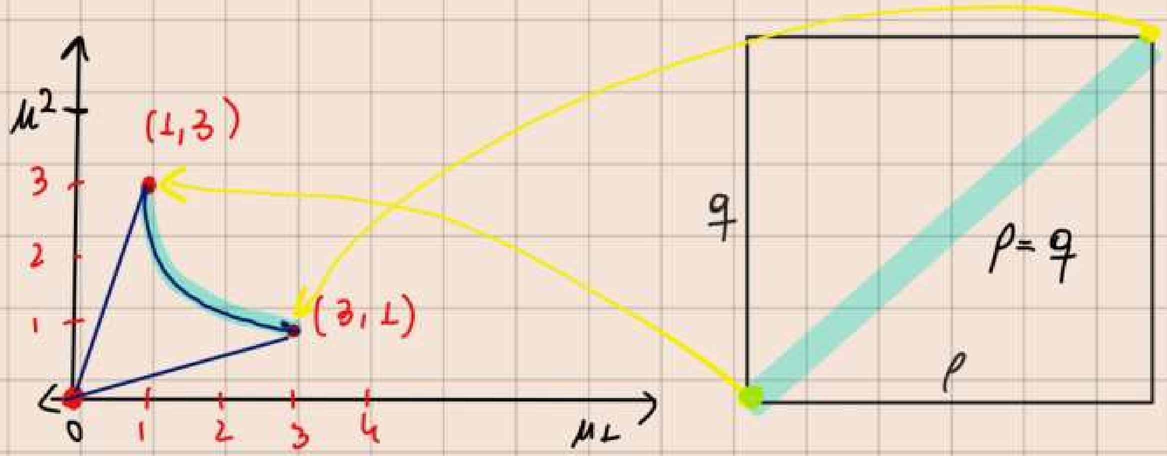
The set of feasible utility profile given that we can play in mixed strategies.

The set of feasible utility profiles (In mixed strategies):



Note that corners just give the same span.

See that (2,2) is not in the triangle. Instead of a Triangle you are going to find the following



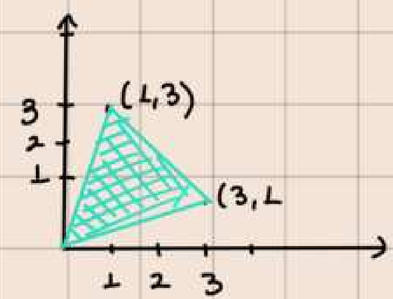
Sector: $\left(pq\,1 + (1-p)(1-q)\,3 \,,\; pq\,3 + (1-q)(1-p)\,1 \right) =$
$\left(p^2 + (1-p)^2\, 3 \,,\; p^2\, 3 + (1-p)^2\, 1 \right)$ for $p \in [0,1]$

CCL: You sometimes get a smaller set than the convex combination of the feasible utilities. You cannot land in (T,L) and (B,R) always. Since you are randomizing you will also see (T,R)(B,L).

DEFINITION: The set $\Delta\left(\bigtimes_{i=1}^{n} A^i \right) \equiv \Delta(A)$ is the set of correlated strategies.

In the Battle of the sexes:

$$u\left(\Delta(A)\right) = \bigcup_{\mu \in \Delta(A)} u(\mu)$$



u is a linear function $\Delta(A)$ is convex then:

$$u(\Delta A) = \left\{ \sum_a \mu(a)\, u(a) \;:\; \mu \in \Delta(A) \right\} = co\left(\left\{ u(a) : a \in A \right\} \right).$$

job). In such situations, it might still be possible for the players to communicate and coordinate with each other. We say that a game is *with communication* if, in addition to the strategy options explicitly specified in the structure of the game, the players have a very wide range of implicit options to communicate with each other. Aumann (1974) showed that the solutions for games with communication may have remarkable properties, even without contracts.

Consider the example in Table 6.4. Without communication, there are three equilibria of this game: (x_1,x_2), which gives the payoff allocation (5,1); (y_1,y_2), which gives the payoff allocation (1,5); and a randomized equilibrium, which gives the expected payoff allocation (2.5,2.5). The best symmetric payoff allocation (4,4) cannot be achieved by the players without binding contracts, because (y_1,x_2) is not an equilibrium. However, even without contracts, communication can allow the players to achieve an expected payoff allocation that is better for both than (2.5,2.5). For example, the players might plan to toss a coin and choose (x_1,x_2) if Heads occurs or (y_1,y_2) if Tails occurs, and thus implement the correlated strategy $0.5[x_1,x_2] + 0.5[y_1,y_2]$. Even though the coin toss has no binding force on the players, such a plan is *self-enforcing*, in the sense that neither player could gain by unilaterally deviating from this plan.

With the help of a *mediator* (that is, a person or machine that can help the players communicate and share information), there is a self-enforcing plan that generates an even better expected payoff allocation $(3\frac{1}{3}, 3\frac{1}{3})$. To be specific, suppose that a mediator randomly recommends strategies to the two players in such a way that each of the pairs (x_1,x_2), (y_1,y_2), and (y_1,x_2) is recommended with probability $\frac{1}{3}$. Suppose also that each player learns only his own recommended strategy from the mediator. Then, even though the mediator's recommendation has no binding force, there is a Nash equilibrium of the transformed game with mediated communication in which both players plan to always obey

Table 6.4 A game in strategic form

	C_2	
C_1	x_2	y_2
x_1	5,1 1/3 1/2	0,0 0
y_1	4,4 1/3	1,5 1/3 1/2

For the coin toss correlated equilibrium example the mediator's recommendations. If player 1 heard a recommendation "y_1" from the mediator, then he would think that player 2 was told to do x_2 or y_2 with equal probability, in which case his expected payoff from y_1 would be as good as from x_1 (2.5 from either strategy). If player 1 heard a recommendation "x_1" from the mediator, then he would know that player 2 was told to do x_2, in which case his best response is x_1. So player 1 would always be willing to obey the mediator if he expected player 2 to obey the mediator, and a similar argument applies to player 2. That is, the players can reach a self-enforcing understanding to each obey the mediator's recommendation when he plans to randomize in this way. Randomizing between (x_1,x_2), (y_1,y_2), and (y_1,x_2) with equal probability gives the expected payoff allocation

$$\tfrac{1}{3}(5,1) + \tfrac{1}{3}(4,4) + \tfrac{1}{3}(1,5) = (3\tfrac{1}{3},\ 3\tfrac{1}{3}).$$

Notice that the implementation of this correlated strategy $\tfrac{1}{3}[x_1,x_2] + \tfrac{1}{3}[y_1,y_2] + \tfrac{1}{3}[y_1,x_2]$ without contracts required that each player get different partial information about the outcome of the mediator's randomization. If player 1 knew when player 2 was told to choose x_2, then player 1 would be unwilling to choose y_1 when it was also recommended to him. So this correlated strategy could not be implemented without some kind of mediation or noisy communication. With only direct unmediated communication in which all players observe anyone's statements or the outcomes of any randomizations, the only self-enforcing plans that the players could implement without contracts would be randomizations among the Nash equilibria of the original game (without communication), like the correlated strategy $.5[x_1,x_2] + .5[y_1,y_2]$ that we discussed earlier. (However, Barany, 1987, and Forges, 1990, have shown that, in any strategic-form and Bayesian game with four or more players, a system of direct unmediated communication between pairs of players can simulate any centralized communication system with a mediator, provided the communication between any pair of players is not directly observable by the other players.)

Consider now what the players could do if they had a bent coin for which player 1 thought that the probability of Heads was 0.9 while player 2 thought that the probability of Heads was 0.1, and these assessments were common knowledge. With this coin, it would be possible to give each player an expected payoff of 4.6 by the following self-enforcing plan: toss the coin and then implement the (x_1,x_2) equilibrium if Heads occurs, and implement the (y_1,y_2) equilibrium otherwise.

Sequence of events

Let $\mu \in \Delta(A)$ a correlated strategy be given.

Assume μ is known to every player.

(1) $a \in A$ is chosen with prob $\mu(a)$

(2) a^i is told to player i, for all i privately.

(3) Players choose the action they prefer, keeping into account the information provided by the "suggested" action a^i.

Definition: $\mu \in \Delta A$ is a correlated equilibrium iff for all $a \in A$ such that $\mu(a) > 0$; every player chooses the suggested action. (Recall player is told a^i)

Take $\mu \in CS$.

(1) $\forall_{i, a^i \in A^i}$ $\mu(a_i)$ = prob of a^i being chosen.

$$\mu(a^i) = \sum_{\bar{a}^i \in A^i} \mu(a^i, \bar{a}^i)$$

(2) If $\mu(a^i) > 0$.

$\mu(\cdot \mid a^i) \in \Delta(\times_{j \neq i} A^i)$ is defined by:

$$\mu(\bar{a}^i / a^i) = \frac{\mu(a^i, a^{-i})}{\mu(a^i)}$$

(3) Given a^i, $\mu(\cdot \mid a^i)$ conditional probability

$$\underset{\mu(\cdot|a_i)}{E}\left[u^i(b^i, \cdot)\right]$$ = expected utility from choosing b^i if told a^i.

$$= \sum_{\bar{a}^i \in A^{-i}} \mu(a^{-i}/a^i)\, u^i(b^i, a^{-i})$$

Example: Utility of player 1

	L	R
T	$\mu(TL)$	$\mu(TR)$
B	$\mu(BL)$	$\mu(BR)$

	L	R
T	a	b
B	c	d

CS:

$$\mu(T) = \mu(TL) + \mu(TR)$$

Player 1 is told to play T.

$\mu(\cdot | T) \in \Delta(A^2)$ if $\mu(T) > 0$, define:

$$\mu(L|T) = \mu(LT) / \mu(T).$$

$$\mu(R|T) = \mu(TR) / \mu(T).$$

Given that T has been suggested, player 1 computes the expected utility given this suggestion of T and B.

$$E_{\mu(\cdot|T)} u^1(T, \cdot) = \mu(L/T)\, a + \mu(R/T)\, b.$$

he has been told T.

$$E_{\mu(\cdot, T)} u^1(B, \cdot) = \mu(L/T)\, c + \mu(R/T)\, d.$$

Why is traffic light a CE?

		Red	Green
		S	G
Red	S	0 bad	1/2 good for 2, ok for 2
Green	G	1/2 good 1, ok 2	0 very bad

you should take your action conditional on that the other has been told to stop.

DEFINITION: μ satisfies IC (Incentive compatibility) if:

$\forall i, a_i$: $\mu(a_i) > 0$ for all $b_i \in A_i$:

$$\sum_{\bar{a}^{i} \in A^{-i}} \mu(a^{-i} / a^{i})\, u^{i}(a^{i}, a^{-i}) \geqslant \sum_{\bar{a}^{i} \in A^{-i}} \mu(\bar{a}^{-i} / a^{i})\, u^{i}(b^{i}, \bar{a}^{-i}) \quad (1)$$

DEFINITION: $\mu \in \Delta(A)$ correlated strategy is a correlated equilibrium if it satisfies IC.

Note that (conditional on $\mu(a^i) > 0$):

$$\sum_{\bar{a}^{i} \in A^{-i}} \mu(a^{-i} / a^{i})\, u^{i}(a^{i}, a^{-i}) \geqslant \sum_{\bar{a}^{i} \in A^{-i}} \mu(\bar{a}^{-i} / a^{i})\, u^{i}(b^{i}, \bar{a}^{-i})$$

$\forall b_i \in A_i$.

can be rewritten using (recall: $\mu(a_i) > 0$).

$\mu(a^{-i}/a^i) \equiv \mu(a^i, a^{-i}) / \mu(a^i)$ as:

$$\sum_{\bar{a}^i \in A^{-i}} \frac{\mu(a^i, a^{-i})}{\mu(a^i)} u^i(a^i, a^{-i}) \geq \sum_{\bar{a}^i \in A^{-i}} \frac{\mu(a^i, \bar{a}^{-i})}{\mu(a^i)} u^i(b^i, \bar{a}^{-i}) \quad \forall b^i \in A_i$$

$$\sum_{\bar{a}^i \in A^{-i}} \mu(a^i, \bar{a}^{-i}) \; u^i(a^i, a^{-i}) \geq \sum_{\bar{a}^i \in A^{-i}} \mu(a^i, a^{-i}) \; u^i(b^i, a^{-i}) \quad \forall b^i \in A_i$$

$$\sum_{\bar{a}^i \in A^{-i}} \mu(a^i, a^{-i}) \left[u^i(a^i, a^{-i}) - u^i(b^i, a^{-i}) \right] \geq 0. \quad (2) \quad \forall b^i \in A_i$$

$\int v(a^i) > 0$

Let v be a another CE then $\forall i \; \forall a_i$, we know that

$$\sum_{a^{-i} \in A^{-i}} v(a^i, a^{-i}) \left[u^i(a^i, a^{-i}) - u^i(b^i, a^{-i}) \right] \geq 0 \quad \forall \; b_i \in A_i$$

<u>Corollary:</u> the set of correlated equilibria is convex and closed.

<u>Proof:</u> Suppose μ and v satisfy (2) and $r \in (0, 1)$ define

$g(a) = r\,\mu(a) + (1-r)\,v(a)$. (Assuming $\mu(a) > 0$, $v(a) > 0$)

Then $g(a)$ satisfies $(2) \Rightarrow (1) \Rightarrow$ CE.

Define $u^i(\mu) \equiv \sum_{a \in A} \mu(a)\, u^i(a)$ and $u(\mu) = (u^i(\mu))_{i \in I}$:

Consider the subset of $\mathbb{R}^n$.

$\{ u(\mu) : \mu \in CE \}$ = set of CE payoffs. $u : CE \to \mathbb{R}^n$

Theorem: The set of CE payoffs is convex and closed.

Take $\mu, \nu \in CE$ and $r \in (0,1)$. Compute $u(r\mu + (1-r)\nu) = r\, u(\mu) + (1-r)\, u(\nu)$ but since $(r\mu + (1-r)\nu) \in CE$ we know that the set CE payoffs is convex. Since $\{\mu \in CE\}$ is closed and u is a continuos function $\{u(\mu) : \mu \in CE\}$ is closed and Bounded.

Take $\hat{s} \in NE$. Consider $Pr_{\hat{s}} \in \Delta(A)$, so is a CS.

What happens if (The mediator is told to follow $\hat{s}$) you apply the mechanism to $Pr_{\hat{s}}$?

		q L	$1-q$ R
p	T	pq	$p(1-q)$
$1-p$	B	$(1-p)q$	$(1-p)(1-q)$

If player 1 is told T:

$$Pr_{\hat{s}}(\cdot \mid T) = \left(pq / (pq + p(1-q)) \,,\; p(1-q) / pq + p(1-q) \right)$$

$$= (\overset{L}{q},\ \overset{R}{1-q})$$

$$Pr_{\hat{s}}(\cdot \mid B) = \left((1-p)\,q \big/ \left[(1-p)q + (1-p)(1-q)\right] ;\ (1-p)(1-q) \big/ \left[(1-p)q + (1-p)(1-q)\right] \right)$$

$$= (\overset{L}{q} ;\ \overset{R}{1-q})$$

If $\hat{s}$ is a mixed strategy profile, then:

$$Pr_{\hat{s}}(\cdot \mid a^i) = Pr_{\hat{s}}(\cdot \mid b^i) \quad \forall \quad a^i, b^i \text{ st } s^i(a^i) > 0 \quad s^i(b^i) > 0.$$

Therefore the optimal choice of action if told a^i is going to be the same that if Told b^i and it is equal to the $BR^i_{s^i}(s^{-i})$.

<u>Theorem</u>: If $\hat{s}$ is a Nash Equilibrium then $Pr_{\hat{s}}$ is a correlated equilibrium.

<u>Corollary</u>: The set of CE is nonempty

<u>Proof</u>: $\{Pr_{\hat{s}} : \hat{s} \in NE\} \neq \emptyset$ and $\{Pr_{\hat{s}} : \hat{s} \in NE\} \subseteq CE$.

Extensive Form Games

Lecture 7

10-02

$I = \{1, \dots, n\}$ set of players

N = nature

$\{1, 2, \dots, n\} \cup \{N\}$ = set of labels.

Example:

$I = \{1, 2\}$, each player has an endowment of (w_1, w_2)

1. N chooses $M \in \{L, H\}$ $\left(M = \begin{array}{cc} H & L \\ 1/2 & 1/2 \end{array} \right)$ $M \in \mathbb{R}_+ \left(\begin{array}{cc} 1 & 2 \\ 1/2 & 1/2 \end{array} \right)$

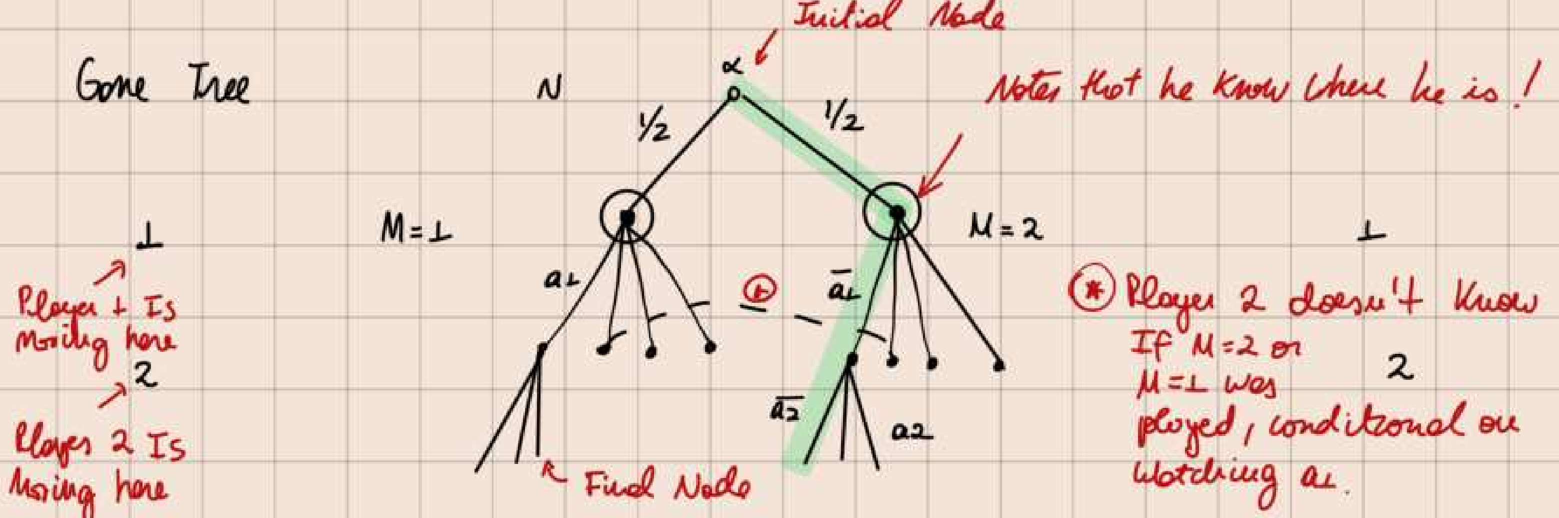
2. M is told to player 1, (P1) (Not to player 1)

3. P1 choose $a^i \in A^i \subset \mathbb{R}$ private.

4. a^1 is told to P2, who chooses $a^2 \in A^2 \subset \mathbb{R}$

5. Game ends, players get $(w^1 - a^1 + a^2,\ w^2 + M a_1 - a^2)$

Game Tree



Suppose we play the green part.

$$\left(W^1 - \bar{a}_1 + \bar{a}_2 \ , \ W^2 + 2\bar{a}_1 - \bar{a}_2\right)$$

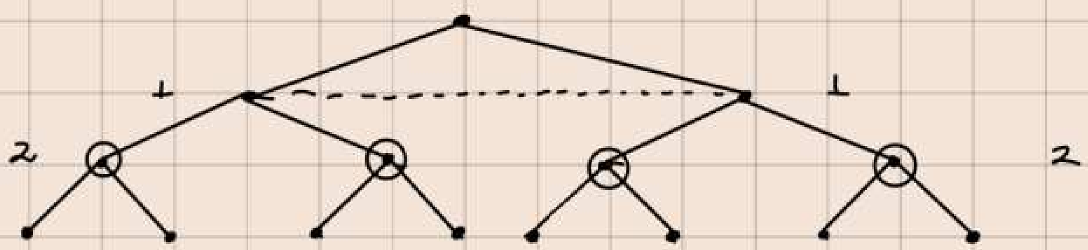
Information set is who knows what at the time he has to move.

DEFINITION: Incomplete information: is where some of the events are not know

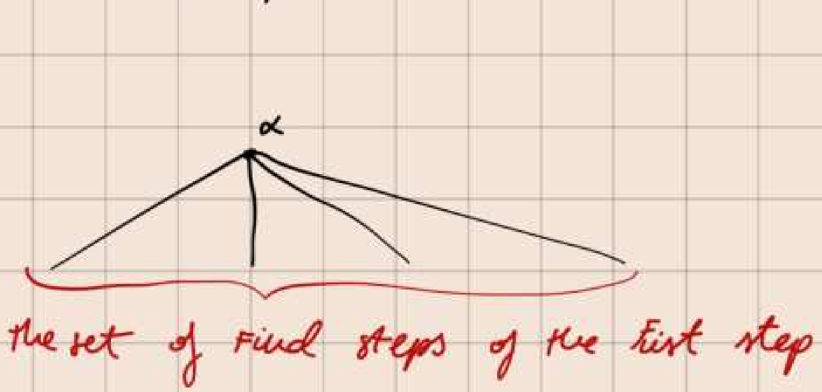
DEFINITION: Imperfect information: a player doesn't know some of the Action of other player

Variation of Example 1: In addition P2 is not told a_1

Variation EX 2: Player 1 Is not told M, P2 is told M and a_1.



The payoffs are the same, the tree structure but not the information sets



Game tree

$I = \{1, \dots, n\} \cup \{N\}$ = set of labels.

DEFINITION: A game tree is the result of 0 set an initial node, (N); α choose a set of sucessors of the initial node

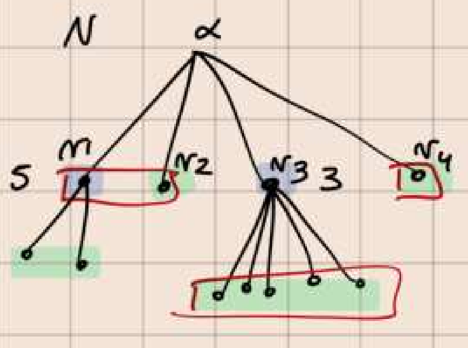
1. Repeat the following a finite number of steps.

   1.1 Pick a final node in the existing game tree. label it by an index I.

   1.2 Add a set of IS of that node

   1.3 start again

Step 1: Initial node given to Nature



Inmediate sucesor

$P(5) = v_1$ $P(3) = v_3$
$X = \{v_1, v_3\}$

Inmediate sucesor v_1

Inmediate sucesors of v_3.

More nodes

I can pick one for example v_1 and assign the Inmde sucesor

Note that only the final nodes are not labeled.

Note: only more nodes are labeled, and all of them are.

This iteration produces:

1. A set of nodes:

$$\underbrace{\{\alpha\}}_{\text{Initial}} \cup \underbrace{X}_{\text{more}} \cup \underbrace{Z}_{\text{Final nodes}} = V$$

2. $X = \bigcup_{i \in I} X_i$, $i \neq j \Rightarrow X_i \cap X_j = \phi$. (one node can belong only to one player X_i = set of nodes labelled "i")

Note:

1. There is a partial order over the set of nodes $>$.

$x > y$ if "x comes before y" iff

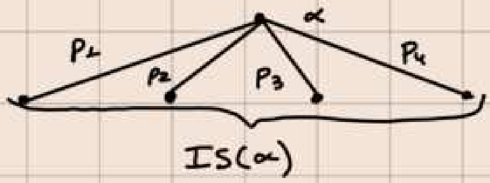
$y \in IS(x) \cup IS(IS(x)) \cup \ldots$

2. **DEFINITION:** For any $y \in$ set of nodes, the path to y is the set $\{x : x > y\}$ is totally ordered by $>$.

DEFINITION: Totally ordered $\forall_{x_1, x_2 \in \{x : x > y\}} \; x^1 > x^2$ or $x^2 > x^1$

3. $\forall_{x \in \text{set of nodes}} \quad \alpha > x$.

1. Probability of Nature's choice:



there is a $p \in \Delta\, IS(\alpha)$ know to all players. In our example

2. Utilities:

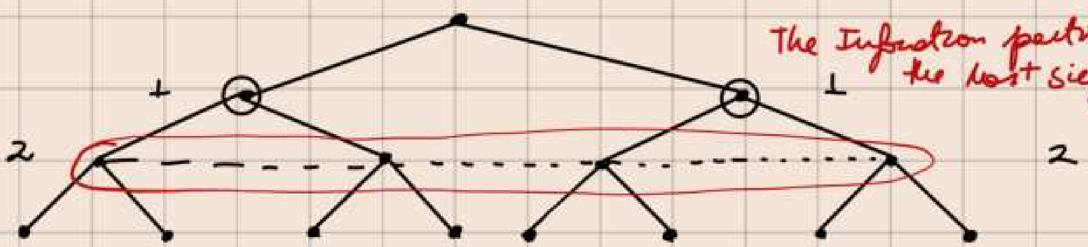
$\forall i \; \exists\, u_i : Z \longrightarrow \mathbb{R}$.

DEFINITION: A game of perfect information happens when all players know all relevant information

the Information sets are partition of the nodes where they move:

The Information partition is the last single ones.

the set of Move Nodes Is the hole set of all Move Nodes.



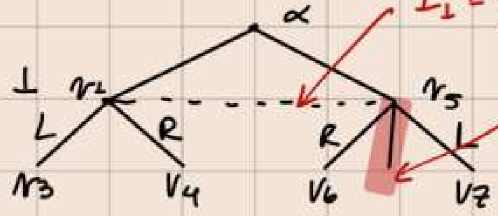
DEFINITION: $X = \bigcup_{i=1}^{n} X_i$, each X_i is a partitioned into subsets

$$X_i = \bigcup_{j=1}^{k_i} I_j^i \,,\quad j \neq k \Rightarrow I_k^i \cap I_j^i = \emptyset$$

Note: For $x, y \in I_j^i$, player i cannot tell where x or y is true

Example: Consider

$I_1^1 = \{v_1, v_5\}$ $\quad C(I_1^1) = \{L, R\}$.



Note that this cannot happen!

$Is(v_1) = \{v_3, v_4\}$ each correspond to a choice of player 1

Assumption on game tree to make sense:

$$\forall_i \ \forall I_j^i \ \forall_{x,y \in I_j^i} \quad \# IS(x) = \# IS(y)$$

DEFINITION: Let $(C(I_j^i))$ be the set of choices available to i at I_j^i

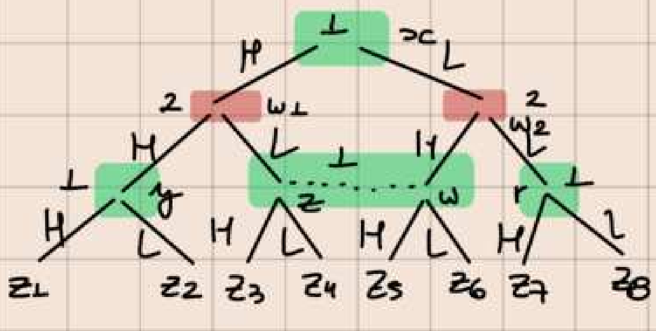
DEFINITION: An EFG is $\big(I, \{N\}, \alpha, X, Z, >, (X_i)_{i \in I}$
$(I_j^i)_{i \in I, j=1,\dots,k^i}, C(I_j^i)_{i \in I, j=1,\dots,k^i}, (u^i)_{i \in I}, p\big)$.

Lecture 8

02-15

Strategies

Example:



$I_1^1 = \{x\}$, $I_2^1 = \{y\}$, $I_3^1 = \{z, w\}$, $I_4^1 \{r\}$.

$I_1^2 = \{w_1\}$, $I_2^2 = \{w_2\}$.

$C(I_1^1) = \{L_1^1, R_1^1\}$

$C(I_2^1) = \{L_1^2, R_1^2\}$

$C(I_3^1) = \{L_1^3, R_1^3\}$

$C(I_4^1) = \{L_1^4, R_1^4\}$

A path of actions for player 1 is a choice out of each of the sets above.

EXAMPLE: The set of pure strategies for P1:

$$S^1 \equiv C(I_1^1) \times C(I_2^1) \times C(I_3^1) \times C(I_4^1).$$

In general;
$$S^i = \bigtimes_{j=1}^{\# I^i} C(I_j^i)$$

change of notation with respect to PART I.

since $\forall_{ij}$ $C(I_j^i)$ is a finite set, S^i is a finite set, so we can denote the set of mixed strategies:

$$\Sigma^i = \Delta(S^i) \quad \text{(this a distribution over all possible action plans)}$$

$$\sigma^i = \left(\sigma(L,L,L,L)\ ;\ \sigma(L,R,L,R)\ ;\ \sigma(R,L,R,L)\ ;\ \dots\ ;\ \sigma(R,R,R,R) \right)$$

Example: There are 16 possible plans (2^4).

Now we have utility functions over the final nodes, we want to rewrite the utility function over the action profiles.

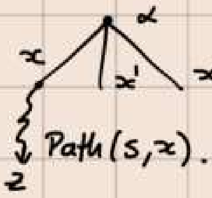
Utilities

$\forall i$; $\quad u^i : Z \longrightarrow \mathbb{R}$.

we want $\forall i$ $\quad U^i : S^1 \times \dots \times S^n \longrightarrow \mathbb{R}$

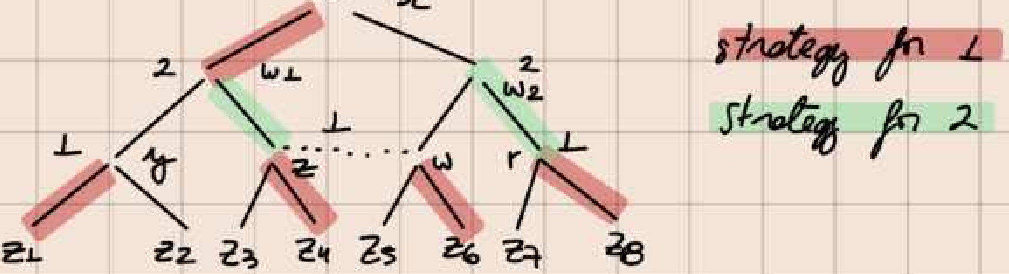
for say $(s_1, s_2, \dots, s_n)$, utility forms from:

① A random element:



(Note that there can be many s leading to z)

Example of strategy:



What's the utility for each player of the path induced by the strategy profile s?

For every $s \in S = \times_{i=1}^{n} S^i$:

$$u^i(s) = \sum_{x \in IS(\alpha)} P(x)\, \mu^i(\text{Path}(s,x))$$

We can Extend this utility as usual to $u^i : \Sigma \to \mathbb{R}$ as:

$$u^i(\sigma) = u^i(\sigma^1, \dots, \sigma^n) \qquad \text{where} \qquad \sigma \in \Sigma = \times_{i=1}^{n} \Sigma_i .$$

$$= \sum_{s \in S} \underbrace{\prod_{j=1}^{n} \sigma^j(s^j)}_{Pr_\sigma(s)} \; u^i(s)$$

We have constructed a normal form game, finite

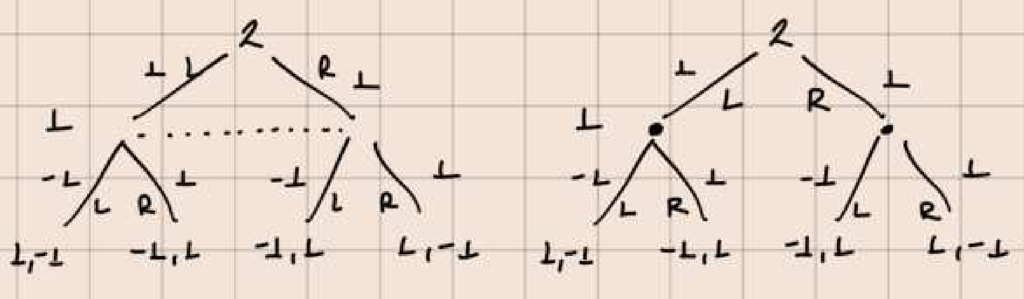
$$(I, (S^i)_{i \in I}, (u^i)_{i \in I})$$

(If we consider the mixed strategys then we are only considering the mixed extension of the NFG)

By Nash's Theorem, an equilibrium of the NFG exists.

DEFINITION: A Nash Equilibrium of an EFG is any NE of the associated NFG.

Matching Pennies:



The Normal form of this game if we don't consider the Information set:

	$L_1^2 L_2^2$	$L_1^2 R_2^2$	$R_1^2 L_2^2$	$R_1^2 R_2^2$
L_1^1				
R_1^1				

$S^1 = \{L, R\}$

$S^2 = \{L, R\} \times \{L, R\}$

<u>Note</u>: The NFG of an EFG depends on the information sets.

Games of Perfect Information

Definition: A game is of PI iff:

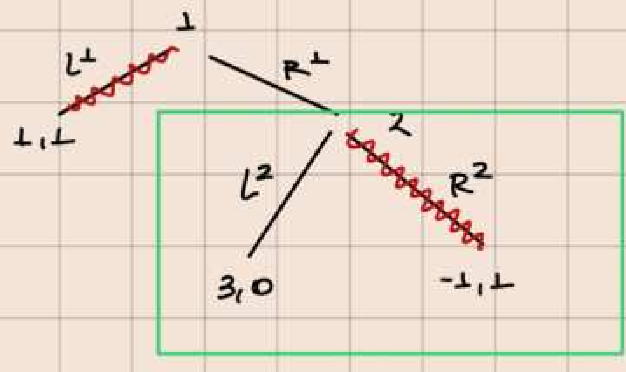
$\forall_i \quad \forall_{I_j^i} \,, \quad I_j^i = \{x\}$ for some node x.

<u>Example</u>: Matching pennies is not a PI game.

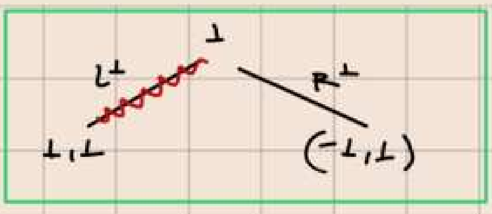
<u>Example</u>: Backwards Induction for the alternative Matching Pennies

Theorem (Zermelo): Games of PI have a NE in pure strategy that can be obtained by backward Induction.

<u>Example Backward Induction:</u>



We can think about:



Following the same idea

$$\boxed{\begin{array}{c} 1 \\ (1,1) \end{array}}$$

We have assign for every agent an action:

	L^2	R^2
L^1	1,1	1,1
R^1	3,0	-1,1

Note that we found the only Nash Equilibrium of the normal form game which is the same as the one BI.

Backwards Induction

<u>Fact</u>: In any finite game of perfect information Γ there exists a node x such that:

$$IS(x) \subseteq Z.$$

<u>Proof</u>: Suppose not x_1, since $IS(x_1) \not\subseteq Z$.

$x_2 \in IS(x_1)$ st $x_2 \notin Z$.

$\vdots$

We can construct an infinite sequence of nodes

$x_1 > x_2 > x_3 > \dots$ in a finite game which contradicts finiteness.

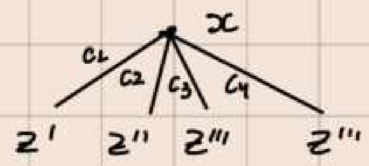
$\square$

We construct a sequence Γ_K of EFG's:

① Find an x in the nodes of Γ_K st $IS(x) \subset Z_{\Gamma_K}$.

② Find the player moving at x find:

$$\hat{c}(x) = \underset{c \in C^i(x)}{\arg\max}\ u^i(z) \qquad \text{where } z = \text{successor of } x$$



③. Delete all $IS(x)$, replace x by a final node st

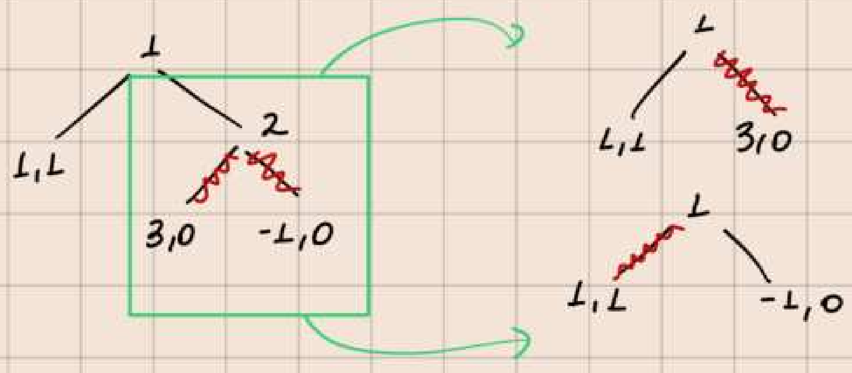
$\forall_j$ $u^j((x, \hat{c}(x)))$ is the utility of j at that new

final node created from following $(x, \hat{c}(x))$.

④ Start Again

This procedure terminates in a finite number of steps, and produces a strategy profile (pure strategies) which is a NE of the EFG.

What happens if we have ties?



		q	$1-q$
		L^2	R^2
p	L^1	1,1	1,1
$1-p$	R^1	3,0	-1,0

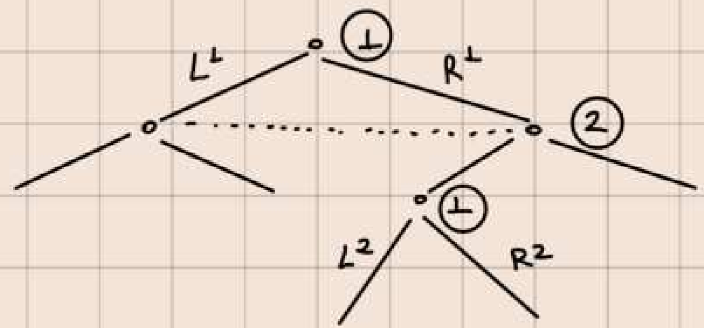
$$\left.\begin{array}{ll} L^1 & 1 \\ R^1 & 3q-(1-q) \end{array}\right\} \text{ if } q = 1/2 \text{ any } p \text{ is a NE.}$$

<u>CCL</u>: Backward Induction only finds 2 NE (But there where more).

Behavioural strategys

We have considered $s^i \in \Delta(S^i)$ (mixed strategies).

Ex ante randomization: player chooses an $s^i \in S^i$ according to $\sigma^i \in \Sigma^i$.



$$S^1 = \{(L^1, L^2), (L^1, R^2), (R^1 L^2), (R^1, R^2)\}$$

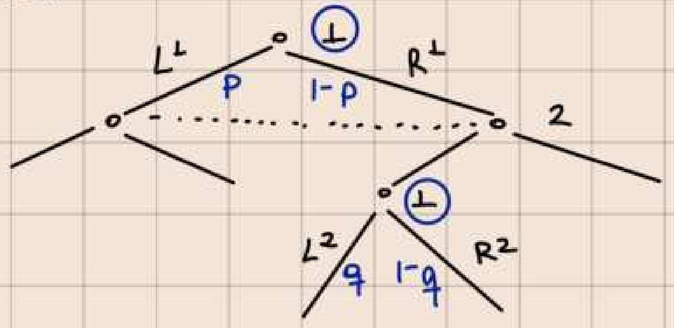
$$\Sigma^1 = \{\sigma \in \Delta^4 : \sigma = (\sigma^1(L^1, L^2); \sigma^2(L^1, R^2); \sigma^3(R^1, L^2); \sigma^4(R^1, R^2)\}$$

$$S^1 = C(I_1^1) \times C(I_2^1)$$

$$\Sigma^1 = \Delta\left(C(I_1^1) \times C(I_2^1)\right)$$

A different randomization

Let P1 choose $(p;q)$ st:



$p \geq 0$, $q \geq 0$. what is the object?

$$(p, 1-p) \in \Delta\left(C(I_1^1)\right) = \Delta\left(\{L^1, R^1\}\right)$$

$$((p,1-p);(q,1-q)) \in \bigtimes_{j=1}^{\#I^1} \Delta\left(C(I_j^1)\right) \quad \text{where } \sigma^i \in \Delta\left(\bigtimes_{j=1}^{\#I^i} C(I_j^i)\right).$$

DEFINITION: The set of behavioral strategies for player i is:

$$B_i = \left\{\beta^i : \beta^i \in \bigtimes_{j=1}^{\#I^i} \Delta\left(C(I_j^i)\right)\right\}$$

you are comitting to following a certain randomization device given that we arrive to certain point.

Questions :

(i) Does it make a difference using B^i or S^i ?

(ii) How do we define NE if we use BS?

<u>The set of behavioral strategies:</u>

Lecture 9

17-02

<u>Recall</u>:

$$B^i = \bigtimes_{j=1}^{\#I_i} \Delta(C_j^i) \quad \text{where} \quad C_j^i \equiv C(I_j^i).$$

$\beta^i = \left(\overset{L^1 \; R^1}{(p, 1-p)} ; \overset{L^2 \; R^2}{(q, 1-q)} \right).$

$\beta^i = (0.2, 0.8, 0.3, 0.7)$

the set of mixed strategies:

$$\Sigma^i = \Delta\left(\bigtimes_{j=1}^{\#I_i} C_j^i \right) = \Delta(S^i)$$

$s^i = \left(\sigma(L^1, R^2), \sigma(L^1, L^2), \sigma(R^1, L^2), \sigma(R^1, R^2) \right).$

$s^i = (0.2, 0.2, 0.2, 0.4)$

The player now has to chose, one way to randomize, we want to compare B^i to Σ^i. Define $\Gamma^i = B^i \cup \Sigma^i$. (this is well defined)

For $\gamma = (\gamma^1, \dots, \gamma^n) \in \bigtimes_{i=1}^{n} \Gamma^i.$

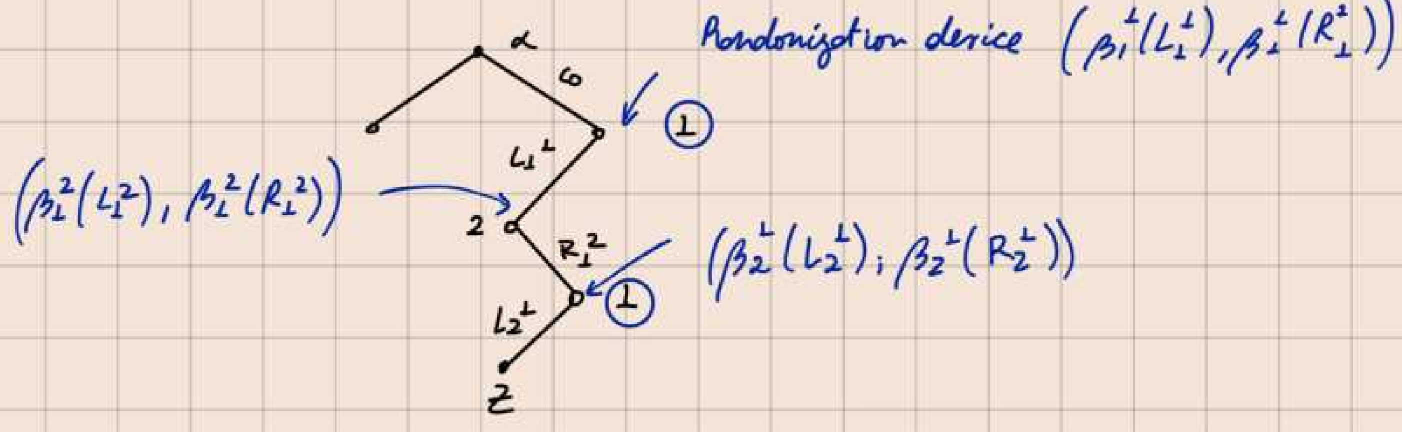
γ^{-i} of others:

$$\max\left(\max_{\beta^i \in B^i} u^i(\beta^i, \gamma^{-i}) , \max_{\sigma^i} u^i(\sigma^i, \gamma^{-i}) \right)$$

Is this the same ? yes, under some conditions.

Definition: $Pr_\gamma(z) \in \Delta$ is the probability of z if $\gamma = (\gamma^1, \dots, \gamma^n)$

Example: How can I reach z?



$$\underset{(\beta^1,\beta^2)}{Pr}(z) = p(c_0) \cdot \beta_1^1(L_1^1) \cdot \beta_1^2(R_1^2) \cdot \beta_2^1(L_2^1)$$

$$\underset{(\sigma^1,\beta^2)}{Pr}(z) = p(c_0) \cdot \sigma^1(L_1^1, L_2^1) \cdot \beta_1^2(R_1^2)$$

$$\underset{(\sigma^1,\sigma^2)}{Pr} = p(c_0) \cdot \sigma^1(L_1^1, L_2^1) \cdot \sigma^2(R_1^2)$$

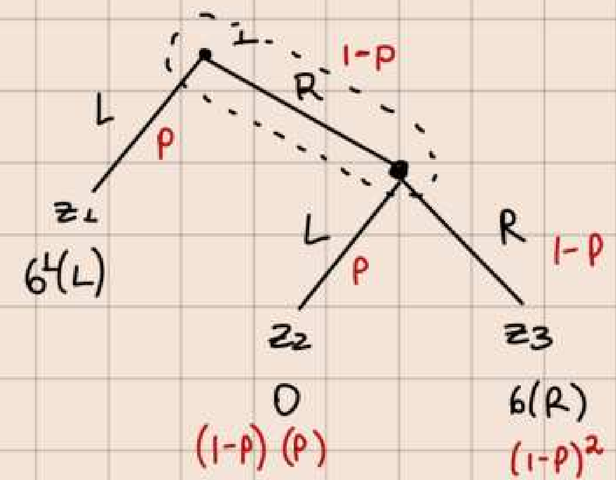
$$\underset{(\beta^1,\sigma^2)}{Pr} = p(c_0) \cdot \beta_1^1(L_1^1) \cdot \sigma^2(R_1^2) \cdot \beta_2^1(L_2^1)$$

Question: For every $\gamma^1, \dots, \gamma^n \equiv \gamma$, to compute $Pr_\gamma(z)$ does it matter whether B^i or Σ^i?

Example: $\Sigma^i \subset B^i$

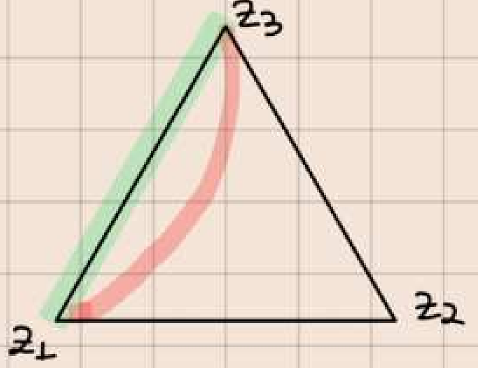
$S^1 = \{L, R\}$ and

$C(I_1^1) = \{L, R\}$.



$$\Sigma^1 = \Delta(\{L, R\}) = \{(\sigma(L), \sigma(R))\}$$

$$\bigcup_{\sigma^1 \in \Sigma^1} Pr_{\sigma^1}(z) = \qquad \text{set.}$$



$$B^1 = \Delta(C) = \{(p, 1-p) : p \in [0, 1]\}$$

$$\bigcup_{\beta^1 \in B^1} Pr_{\beta^1} = \qquad \text{set.}$$

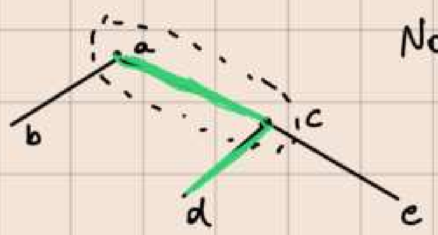
" Mixed strategy. I flip a coin and then choose L or R.
Behavioral strategy. In every turn I will flip a coin."

Now we are going to impose certain conditions, that makes both ways ways of randomizing equal

They are the same if Agents remember their previous actions

DEFINITION: An extensive form game is linear if every path intersects any information set at most once for every player.

Example 1: not linear hence not perfect recall



Note:

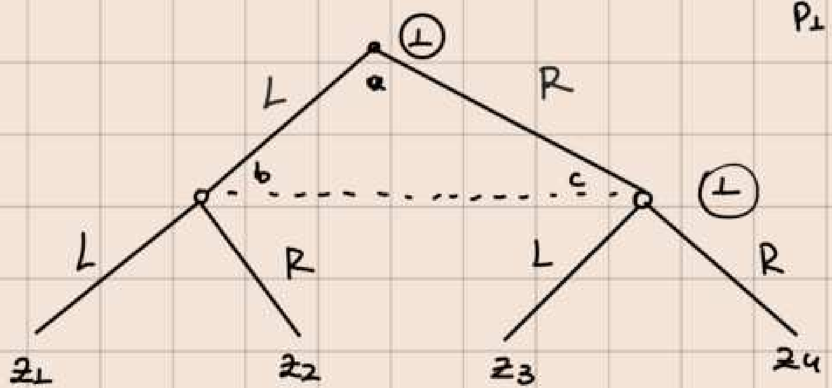
$$\left.\begin{aligned} P_1 &= a \to b \\ P_2 &= a \to c \to d \\ P_3 &= a \to c \to e \end{aligned}\right\} \text{paths.}$$

$I^1 = \{a, c\}$

$P_2 \cap I^1 = \{a, c\}$!

Then is not linear.

Example 2: Linear but Not perfect recall.



P_1 : $a \to b \to z_1$
$a \to b \to z_2$
$a \to c \to z_3$
$a \to c \to z_4$

$I_1^1 = \{a\}$
$I_2^1 = \{b, c\}$

This is a linear game but not perfect recall

DEFINITION: A game is of perfect recall if the following:

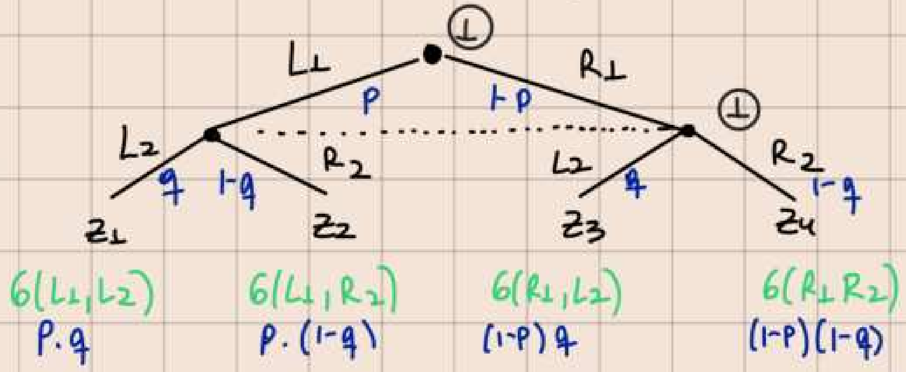
$\exists\ x, y_1, y_2 \quad y_1 \neq y_2,\ y_1, y_2 \in I_2^i,\ x \in I_1^i$ st:
y_1 follows x after an action $c(x, y^1)$ of i, and y^2 does not.

cannot happen.

Why do we want to exclude games which are not perfect recall?

Example: $B_i \subset \Sigma^i$.

I cannot mimic mixed strategy with behavioural strategies



$S^1 = \{L_1, R_1\} \times \{L_2, R_2\}$

$\Delta(S^1) = \{ (\sigma(L_1, R_2), \sigma(L_1, R_2), \sigma(R_1, L_2), \sigma(R_1, R_2) \}$

$$\bigcup_{\sigma^1 \in \Sigma} P_{\sigma^1} = \Delta(Z)$$

Can you get $\begin{Bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 1/2 \end{Bmatrix}$ You can achieve with mixed strategies the set of probability distributions over the final nodes but not BS.

$B^1 = \left(\Delta(\{L_1, R_1\}) \times \Delta(\{L_2, R_2\})\right)$

$\beta^1 = \left((p, 1-p), (q, 1-q)\right)$

Can you get anything you want?

$$\bigcup_{\beta^1 \in B^1} P_{r_{B^1}} = \underset{\text{mixing}}{\text{independent}} \neq \Delta(Z).$$

Note that the set in $\Delta(Z)$ that player 1 can get with mixed strategies is larger. If we recall the drunk driver example, then it was the opposite in that case.

Equivalent strategies

Definition: Take γ^{-i} as given. We say $\gamma^i \sim \tilde{\gamma}^i$ against γ^{-i} if:

$$\Pr_{(\gamma^i, \gamma^{-i})}(z) = \Pr_{(\tilde{\gamma}^i ; \gamma^{-i})}(z) \quad \forall\; z \in Z.$$

We say that γ^i is equivalent to $\tilde{\gamma}_i$ given γ^{-i}

Note: they give the same expected utility.

Question: $\forall\, \beta^i \in B^i$, $(\beta^i, \gamma^{-i}) \Rightarrow \Pr_{(\beta^i, \gamma^{-i})}$: ? $\exists_{\sigma^i \in \Sigma^i}$ st

$$\Pr_{(\sigma^i, \gamma^{-i})}(z) = \Pr_{(\beta^i, \gamma^{-i})}(z)$$

Theorem: In linear games, for every γ^{-i} and every $\beta^i \in B^i$ there is a $\sigma^i \in \Sigma^i$ such that:

$$\Pr(\beta^i, \gamma^{-i}) = \Pr(\sigma^i, \gamma^{-i}) \tag{1}$$

Theorem: A game of perfect recall is a linear game.

Theorem (Kuhn): In a game of perfect recall, for every γ^{-i}:

(i) for every $\beta^i \in B^i$ there is a $\sigma^i \in \Sigma^i$ st:

$$\Pr(\beta^i, \gamma^{-i}) = \Pr(\sigma^i, \gamma^{-i})$$

(ii) for every $\sigma^i \in \Sigma^i$ there is a $\beta^i \in B^i$ st:

$$\Pr(\sigma^i, \gamma^{-i}) = \Pr(\beta^i, \gamma^{-i})$$

Bipul's EFG Notes: $\forall \beta_i, \exists \sigma_i$ Equivalent.

Proof of (1): Let $\beta^i(A^i(I_K^i))$ denote the probability distribution over actions in the information set. $\beta^i(a_i(I_K^i))$ the probability assign to action a_i.

Define the pure strategy s^i: Let $s^i(I_K^i)$ be the action taken at information set I_K^i under the strategy s^i.

Define:

$$\sigma^i_{\beta_i}(s^i) = \prod_{j=1}^{I_K^i} \beta_j^i\left(s^i(I_j^i)\right) \quad \forall \; s^i \in S^i.$$

Now we need to show that the probabilities are equivalent:

$$P(\sigma^i, \gamma^{-i}) = P(\beta^i, \gamma^{-i})$$

If no actions are needed to be taken by i given what the other players play this holds directly.

Let $\tilde{a}^i_{I_K^i}(z)$ be the action needed to be taken at information set I_K^i by player i, to arrive at z given what the other players are playing.

Denote $\tilde{I}^i(z) = \{ I_K^i \mid \text{Path}(z) \wedge I_K^i \neq \phi \}$. Strange not to have into account what others players do.

$$Pr_{\beta_i}(z) = \prod_{\tilde{I}_K^i(z)} \beta_j^i\left(\tilde{a}_{K^i}(z)\right)$$

Note that $Pr_{\beta_i}(z)$ will depend on what the other players are playing. For simplicity we are assuming they are playing degenerate BS.

Let $\tilde{S}^i(z)$ be the set of pure strategies leading to z.

It must be the case that:

This part needs linearity of games

$$\forall \; s_i \in \tilde{S}_i(z), \quad s^i(I_K{}^i) = \tilde{a}^i_{I_{K^i}}(z) \quad \forall \quad I_K{}^i \in \tilde{I}^i.$$

$$\mathbb{P}_6(z) = \sum_{s_i \in \tilde{S}_i(z)} 6(s_i) = \sum_{s^i \in \tilde{S}_i(z)} \left[\prod_{i=1}^{I_K} \beta_i(\overset{s_i(I_i{}^k)}{I_i^k}) \right]$$

$$= \sum_{s^i \in S_i(z)} \left(\prod_{i=1}^{\tilde{I}^i} \beta^i(\overset{s_i(I_i{}^k)}{I_i{}^k}) \cdot \prod_{i=1}^{\tilde{I}_c^i} \beta^i(\overset{s_i(I_i k)}{I_i{}^k}) \right)$$

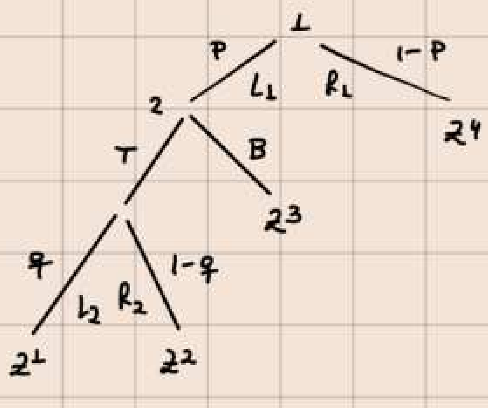
$$= \sum_{s_i \in S_i(z)} \prod_{i=1}^{\tilde{I}^i} \beta_i(a_k^i(z)) \cdot \prod_{i=1}^{I_c^i} \beta^i(\overset{s_i(I_i k)}{I_i{}^k})$$

$$= \prod_{i=1}^{I^i} \beta_i(a_k^i(z)) \underbrace{\sum_{s^i \in S_i(z)} \prod_{i=1}^{I_c^i} \beta^i(\overset{s_i(I_i k)}{I_i{}^k})}_{=1}$$

22-02

Lecture 10

(i) Take $\beta^i \in B^i$:



We want a mixed strategy $\sigma^1(\ ;\beta^1)$ that takes us to the same nodes with the same probability.

$$\sigma^1(L_1 L_2) = pq$$
$$\sigma^1(L_1 R_2) = p(1-q)$$
$$\sigma^1(R_1 L_2) = (1-p)q$$
$$\sigma^1(R_1 R_2) = (1-p)(1-q)$$

	z_1	z_2	z_3	z_4
T	pq	$p(1-q)$	0	$1-p$
B	0	0	p	$1-p$

<u>The general algorithm:</u> (You give a behavioural strategy I can construct a mixed str)

$$\sigma^1(L_1, L_2 \mid \beta_1) = p \cdot q = \beta_1^1(L_1)\, \beta_2^1(L_2)$$

In general for any behavioural β_i we can construct an equivalent mixed strategy $\sigma^i(\cdot, \beta^i)$ as follows:

$$\sigma^i(s^i;\beta^i) = \prod_{j=1}^{\# I^i} \beta_j^i\big(\underbrace{s^i(c_j^i)}\big)$$

this is a pure strategy it tells you what to do at each information set.

$s^i = (s_1^i, \dots, s_j^i, \dots, s_{\# I^i}^i)$ pure strategy profile.

what if $\sigma^1(R_1, L_2) = x$ where $x + y = 1 - p$.

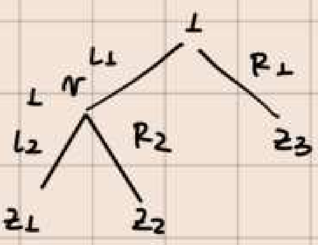
$\sigma^1(R_1, R_2) = y$

$\sigma^1(L_1, L_2) = pq$

$\sigma^1(L_1, R_2) = p(1-q)$

This mixed strategy is equivalent to the behavioral strategy but not unique (Remember that equivalent means that given γ^{-i} it places same prob on final nodes)

(ii) let σ^i be given, we want to construct a behavioural strategy such that they are equivalent. Start from the following simple example:



$$\sigma^1 = \left(\sigma^1(L_1, L_2);\ \sigma^1(L_1, R_2);\ \sigma^1(R_1, L_2);\ \sigma^1(R_1, R_2)\right)$$

$Pr_{\sigma^1}(z_1) = \sigma^1(L_1, L_2)$

$Pr_{\sigma^1}(z_2) = \sigma^1(L_1, R_2)$

$Pr_{\sigma^1}(z_3) = \sigma^1(R_1)$ where $\sigma^1(R_1) = \sigma^1(R_1, L_2) + \sigma^1(R_1, R_2)$

$Pr_{\sigma^1}(v) = \sigma^1(L_1)$ where $\sigma^1(L_1) = \sigma^1(L_1, R_1) + \sigma^1(L_1, R_2)$

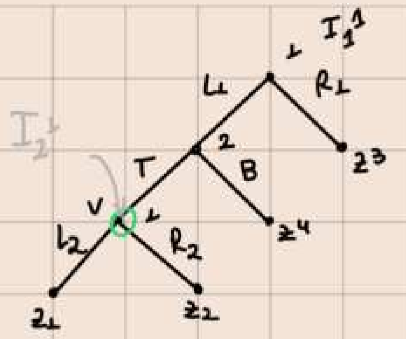
$$\begin{aligned} Pr_{\sigma^1}(z_1) &= \sigma^1(L_1, L_2) \\ &= Pr_{\sigma^1}(z_1 \mid v)\ Pr_{\sigma^1}(v) \\ &= Pr_{\sigma^1}(z_1 \mid v)\ \sigma^1(L_1) \end{aligned}$$ where $\sigma^1(L_1) \equiv \beta_1^1(L^1; \sigma^1)$

We need to produce: $\left(\beta_1^1(\cdot\,; \sigma^1)\ ;\ \beta_2^1(\cdot\,, \sigma^1)\right)$

$\beta_1^1(L_1, \sigma^1) = \sigma^1(L^1)$

$\beta_2^1(L_2, \sigma^1) = Pr_{\sigma^1}(z \mid v) = \dfrac{\sigma(L_1, L_2)}{\sigma(L_1)}$

Example 2:



o Is a node but it is also an information set. I_2^1

$\Pr_{\sigma^1}(v)$ this by itself is meaningless, but then we change a little bit to:

$$\left.\begin{aligned} \Pr_{(\sigma^1, T)}(v) &= \sigma^1(L_1) \\ \Pr_{(\sigma^1, B)} &= 0 \end{aligned}\right\}$$ I cannot match If I don't have the strategy of other players. (Look at the definition of equivalent).

Given I_2^1: (L_1, L_2) (L_1, R_2) can take the path to I_2^1 (If 2 chooses T). therefore $(L_1, L_2), (L_1, R_2) \in \mathrm{Rel}(I_2^1)$.

$(R_1, L_2), (R_1, R_2) \notin \mathrm{Rel}(I_2^1)$.

Definition: For $\forall i$, $\forall (I_j^i)$ of a^i, the relevant set:

$$\mathrm{Rel}(I_j^i) \equiv \{ s^i \in S^i : \exists_{\bar{s}^i \in \bar{S}^i} \text{ path } (s^i, \bar{s}^i) \cap I_j^i \neq \phi \}$$

where $\mathrm{Rel}(I_j^i) \subseteq S^i$. If the other player might play something that

For the general case:

$$\mathrm{Rel}(I_2^1) = \{ (L_1, L_2) ; (L_1, R_2) \}$$

$$\sigma^1(L_1) = \sum_{\{s^i \in \mathrm{Rel}(I_2^1)\}} \sigma^1(s^i)$$

$$\sigma^1(L_1, L_2) = \sum_{\{s^i \in Rel(I_2^1) : s^i(I_2^1) = L^2\}} \sigma^1(s^i).$$

$$\beta_2^1(L_2 ; \sigma^1) = \frac{\sigma^1(L_1, L_2)}{\sigma(L_1)}$$

the General Algorith: For player i

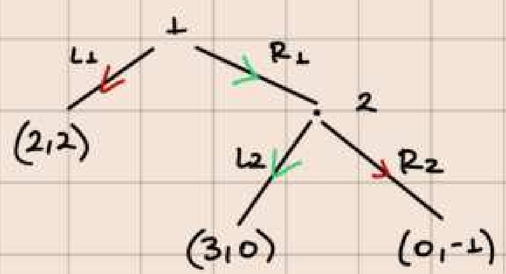
$$\beta_j^i(c_j^i ; \sigma^i) = \frac{\sum_{\{s^i \in Rel(I_j^i) : s^i(I_j^i) = c_j^i\}} \sigma^i(s^i)}{\sum_{\{s^i \in Rel(I_j^i)\}} \sigma^i(s^i)}$$

for $c_j^i \in C(I_j^i)$

$$= \text{arbitrary if } \sum_{\{s^i \in Rel(I_j^i)\}} \sigma^i(s^i) = 0$$

∎

Refinements of NE



	L_2	R_2
L_1	2,2	2,2
R_1	3,0	0,-1

BI gives $\{(R_1, L_2)\}$. Look at the NE $\{(L_1, R_2)\}$ this will never hold since player 2, would never choose the action preestablish.

We are going to restrict ourselves to subgames to look for requirements.

Subgame perfection

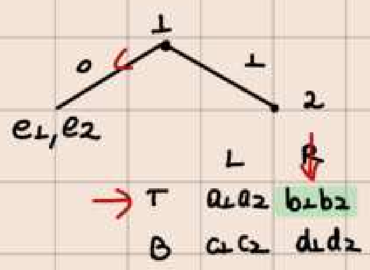
1. Take all NE
2. Take "subgames"
3. check for each candidate NE_m whether it is still a NE in the subgame
4. the subset of NE that passes all these tests are called subgame perfect equilibrium.

		L	R
0	T	e_1,e_2	e_1e_2
0	B	e_1,e_2	e_1e_2
1	T	a_1a_2	b_1b_2
1	B	c_1,c_2	d_1d_2

The Nash equilibrium that doesn't make sense, since Agent 2 is playing a strictly dominated strategy.

Why is it different to BI?

Example:



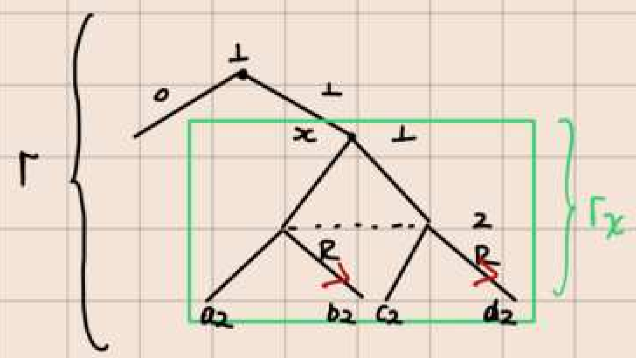
Assume: $a_2 > b_2$

$c_2 > d_2$

$b_1 > d_1$

$b_1 < e_1$

Backward induction cannot even be applied this is not a game of perfect Information.

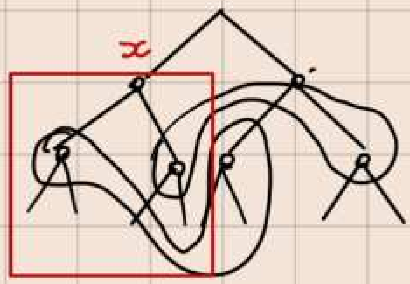


The tree in the square is a EFG. Since:

$a_2 > b_2 \qquad c_2 > b_2$

The Action R of player 2 in Γ_x gives less than L no matter what player 1 does in Γ_x. So $(0, T, R)$ is not a Subgame perfect Eq.

Let $x \in$ set of moves nodes in an EFG Γ $\{x\} = I_j^i$ for some i.



You cannot split information sets. The original game is the subgame.

Lecture 11

<u>subgame perfectation</u>: Elimination of Different NE. 24-02

Definition: No splitting of information sets at x; $\forall i \; \forall_{j=1,\dots,\#i}$

N_x = set of successors of x.

We require $\quad I_j^i \cap N_x = I_j^i \qquad I_j^i \cap N_x = \emptyset$

Definition: (subgame at x) For any $x \in X$ (the set of moves nodes) of an EFG Γ such that "no splitting condition" holds, Γ_x (subgame at x) is the EFG defined by the restriction to N_x;

$$\Gamma_x = \left(I, N_x, X_x, Z_x, \dots, u|_{Z_x} \right).$$

$$X_x \equiv X \cap N_x \qquad Z_x \equiv Z \cap N_x$$

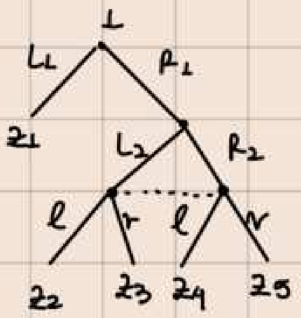
Note the information sets of Γ_x are those who have nodes entirely in N_x.

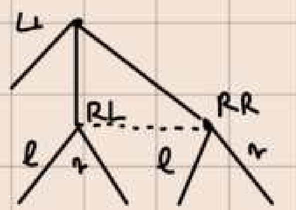
Definition: A vector of behavioral strategy profiles $\hat{\beta}$ is a subgame perfect equilibrium if the restriction of $\hat{\beta}$ to Γ_x for any subgame Γ_x is a nash equilibrium of Γ_x.

Problem: Do SPE exist? They do exist for perfect recall games.

If PR then $\beta_i \Leftrightarrow s_i$. We know for any extensive form game there exists a NFG representation. $\exists$ NE and then we construct β_i accordingly.

Sequential Equilibra





	ℓ	r
L_1L_2	z_1	z_1
L_1R_2	z_1	z_1
R_1L_2	z_2	z_3
R_1R_2	z_4	z_5

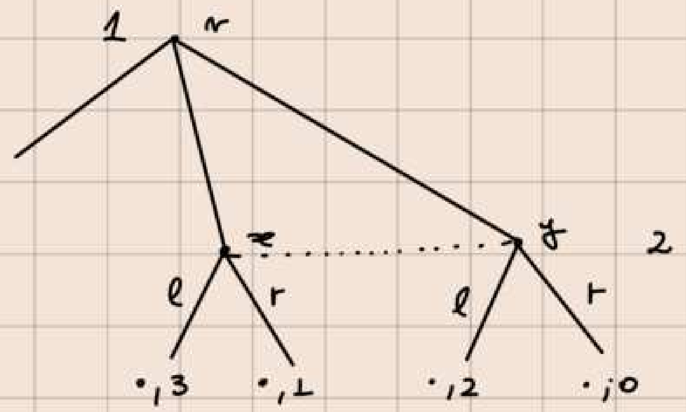
	ℓ	r
L_1	z_1	r_1
RL	z_1	z_3
RR	z_4	z_5

An Equilibrium here is no subgame perfect Because the subgame is not the whole game

This equilibrium is a subgame perfect Equilibra Because the subgame is the whole game

Problem: This Is kind of the same game but it really is not.

Example:



1. $(\mu(x); \mu(y))$ beliefs of 2 on $\{x, y\}$.

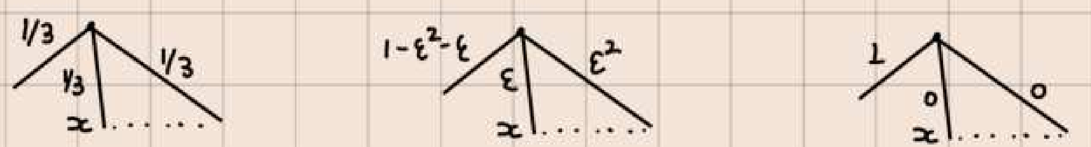
2. For the given μ:

$\ell \to E_\mu u^2$
$r \to E_\mu u^2$

For any μ, $E_\mu u^2(\ell) > E_\mu u^2(r)$

so no matter what player 2 thinks of the odds of x vs y, r is worst than ℓ.

Idea of what's coming: A behavioural strategy of P1 induces a belief on P2



3 cases: The belief of being on x conditional on β_1 is $1/2$
the belief of being at x conditional on β_1 is $\varepsilon / \varepsilon+\varepsilon^2$
the belief of being at x conditional on β_1 is anything

(μ, β) First is what you believe and then what you do.

Definition: Sequential Equilibrium is a pair of (μ, β) such that:

(i) μ is induced by β. ie; μ is the limit of $\mu(\cdot, \beta\alpha)$, $\beta\alpha \to \beta$ for $\beta\alpha$ fully mixed.

(ii) β is sequentially rational given μ.

Definition: A behavioural strategy of i is fully mixed if $\forall_{i=1,\dots \# I^i}$

$$\forall_{c \in C_j^i} \quad \beta_j^i(c_j^i) > 0.$$

Fact: If $\beta \in \bigtimes_{i=1}^{n} B^i$ is fully mixed then for any node x will occur with positive probability.

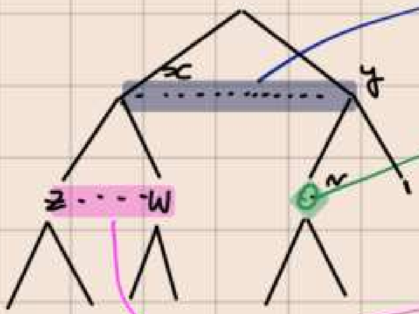
Fact: If $\beta \in \bigtimes_{i=1}^{n} B^i$ is fully mixed, we can apply Bayes Rules and compute $\forall i$ $\forall_{j=1,\dots,\# I^i}$ $\forall_{x \in I_j^i}$:

$$Pr(x_j, \beta, I_j^i) > 0$$

Then we can calculate the posterior on X as:

$$\mu(x; \beta; I_j^i) = \frac{Pr(x, \beta, I_j^i)}{\sum_{y \in I_j^i} Pr(y; \beta, I_j^i)}$$

For any β (forget about the information set) you can get a set of beliefs:



Same Idea applies for this information set

If I tell you are here then the prob should be 1

If I tell you are in this information set, you can tell me what is the prob of being in z or w.

Definition: A system of beliefs is:

$$\mu : X \longrightarrow [0,1] \text{ such that:}$$

$$\forall i \ \forall j \quad \sum_{x \in I_j^i} \mu(x) = 1.$$

$\left(\mu(x,\beta)\right)_{x \in X}$ is a system of beliefs.

Fact: If $\beta_\alpha \xrightarrow[\alpha \to +\infty]{} \beta$ (the behavioral strategy for each choice converges)

$\mu(\cdot, \beta_\alpha) \longrightarrow \mu(\cdot, \beta)$ for some $\mu(\cdot, \beta)$, $\alpha \to +\infty$ and $\mu(\cdot, \beta)$ is a system of beliefs.

Example:

$\mu(x, \beta_\epsilon)$

odd days
even days

convergent subsequences

$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$

Definition : A system of beliefs μ is induced by a sequence of behavioural strategy profiles if $\{ \beta\alpha : \alpha = 1, \dots \}$ if :

$$\mu(\cdot, \beta\alpha) \longrightarrow \mu(\cdot, \beta) \quad \text{when} \quad \beta\alpha \longrightarrow \beta.$$

Lecture 12

Assessment = pair (β, μ)

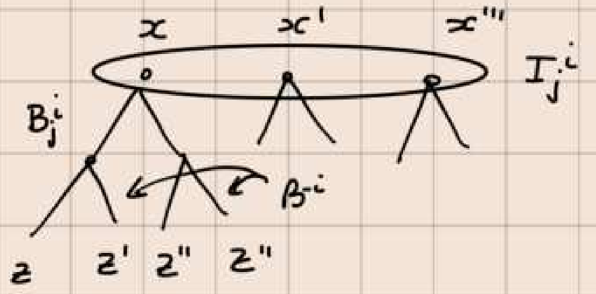
03-01

Definition: (β, μ) is consistent if μ is Induced by β as in the previous definition

Take μ given, β^{-i} also given:

For every x nose node of player i

$Pr(z ; \beta^i, \beta^{-i}, x)$ = probability of z if we start at x and follow (β^i, β^{-i}). (we have not require that beta here is fully mixed).



when you take into account the behavioural strategy at the infomation set, we need to take into account the effect on the other nodes in the information set.

To evaluate a BS β^i at I_j^i against β^{-i} we use μ.

By definition it needs to sum once we take into account all x in the information set.

$$V^i(I_j^i ; \beta^i, \beta^{-i}) = \sum_{\substack{x \in I_j^i \\ z \in Z^i}} \mu(x) \Pr(z ; \beta^i, \beta^{-i}, x) \; u^i(z).$$

given that he is choosing β^i others are in β^{-i} and he has beliefs μ

Is the expected value from β^i given μ and β^{-i}.

Maybe the information set was never reached! But can still compute this

Definition: An assessment (β, μ) is sequentially rational iff

$\forall i \, \forall j$: (At every information set, for all players)

$$\forall_{\gamma^i \in \beta^i} \quad V(I_j^i, \beta^i, \beta^{-i}) \geq V(I_j^i, \gamma^i, \beta^{-i}) \qquad (\star)$$

Definition: An assessment (β, μ) is a sequential equilibrium iff it is consistent and sequentially rational

Clarifications on sequential equilibra:

Definition: A sequential equilibrio is given by (β, μ) st:

1. (β, μ) is sequentionally rational:

$$\forall i \forall j: \quad V^i(I_j^i, \beta_i, \beta^{-i}) \geq V(I_j^i, \gamma_i, \beta^{-i}) \quad \forall\ \gamma_i \in B_i.$$

where $V^i(I_j^i, \beta_i, \beta^{-i}) = \sum_{\substack{x \in I_j^i \\ z}} \mu(x) P(z, \beta_i, \beta^{-i}, x)\ u(z)$

2. μ is consistent

i μ is derived from Baye's rule.
ii. $\exists\ (\beta_n, \mu_n)$ st $\mu_n \to \mu$ $\beta_n \to \beta$ where β_n is fuly mixed.

$P(z, \beta^i, \beta^{-i}, x)$ The probability of landing in z, given that we are in x and we are going to follow β^i and β^{-i} from then on.

Given that the agent i, believes that he is in each node of I_j^i according to $\mu(x)$, we compute the value of following β^i given β^{-i}.

Sequential rationality requires that optimal behaviour strategy are played at each information set, regardless on whether the information set is played or not.

Perfect Extensive form equilibria

Recall: PE for NFG

We can define the analog for extensive form games in two ways.

(1) Perturbing BS (Trembling hand)

(2) Perturbed MS of NFG (Perfect Equilibria of the normal form)

The two definitions are not equivalent even in games of perfect recall (In spite of Kuhn's theorem).

In games where each player has one information set they are.

Fully ε-mixed

Definition: for $\varepsilon > 0$, for all i:

$$B_\varepsilon^i = \{ \beta^i \in B^i : \forall_{j=1,\dots,\#I^i} \ \forall_{c \in C_j^i} \ \beta_{\varepsilon j}^i(c) \geq \varepsilon \}$$

("each action has to be chosen at least with probability ε").

Definition: Γ_ϵ is the perturbed EFG where player have to choose in B_ϵ^i, for $\epsilon > 0$.

Definition: A Nash Equilibrium Γ_ϵ is a $\hat{\beta}_\epsilon$ such that $\forall i$:

$$\hat{\beta}_\epsilon^i \in BR^i_{B_\epsilon^i}(\hat{\beta}^{-i})$$

This is not required in our sequential Equilibrium definition

Theorem: For every $\epsilon > 0$ there is a NE of Γ_ϵ.

Theorem: Possibly on a subsequence as $\epsilon \longrightarrow 0$ there is $\hat{\beta}_\epsilon \in NE(\Gamma_\epsilon)$ such that:

$$\hat{\beta}_\epsilon \longrightarrow \beta$$

for some $\beta \in B$.

(since it lives in a compact set we know that the sequence has a limit point).

Definition: Any limit β of a sequence $\{\beta_\epsilon\}$ where $\beta_\epsilon \in NE(\Gamma_\epsilon)$ is called a perfect equilibria of the extensive form.

Theorem: Sequential Equilibrium exists given a Perfect Equilibria

Proof: A Perfect Eq $\hat{\beta}$ can be completed with $\hat{\mu}$ to give a sequential Eq

Take $\hat{\beta}$ a perfect equilibria:

$$\exists_{\epsilon \to 0} \quad \beta_\epsilon \in NE(\Gamma_\epsilon) \quad \beta_\epsilon \to \hat{\beta}$$

$\mu(\cdot, \hat{\beta_\epsilon})$ is well defined. $\exists \mu$: $\quad \mu(\cdot, \hat{\beta_\epsilon}) \longrightarrow \hat{\mu}$.

Possibly on a subsequence:

Claim: $(\hat{\beta}, \hat{\mu})$ is a consistent assesment (It's also seq rational, think about it and prove it). Because the inequality (☆) holds $\forall i \ \forall I_j^i$ at $\hat{\beta}_\varepsilon$ and the inequality is preserved in the limit.

Games of perfect Recall (GPR)

(1) Perfect (TH) $\Longrightarrow$ Seq Eq

(2) Seq Eq $\Longrightarrow$ Perfect

In games of perfect Information:

(3) Subgame perfect (SPE) $\Longleftrightarrow$ Seq. Equilibria.

(4) GPR have a SPE (This is false in not perfect recall games).