FINAL PREP

Recall that the set of behavioural strategies was defined as:

$$B^i = \mathop{\times}_{j=1}^{\#I^i} \Delta(C_j^i) \quad \text{where} \quad C_j^i \equiv c(I_j^i)$$

the set of mixed strategies was defined as:

$$\Sigma^i = \Delta\left(\mathop{\times}_{j=1}^{\#I^i} C_j^i\right)$$

This should be clear, If it is not take time to read the pevious recitation notes.

Questions: Is it the same for the agent to choose strategies from Σ^i, rather than from B^i? If it is not, can we provide conditions such that it is?

Definition: $\Gamma^i \equiv B^i \cup \Sigma^i$, $\quad \gamma = (\gamma^1, \dots, \gamma^I) \in \mathop{\times}_{i=1}^{I} \Gamma^i \equiv \Gamma$.

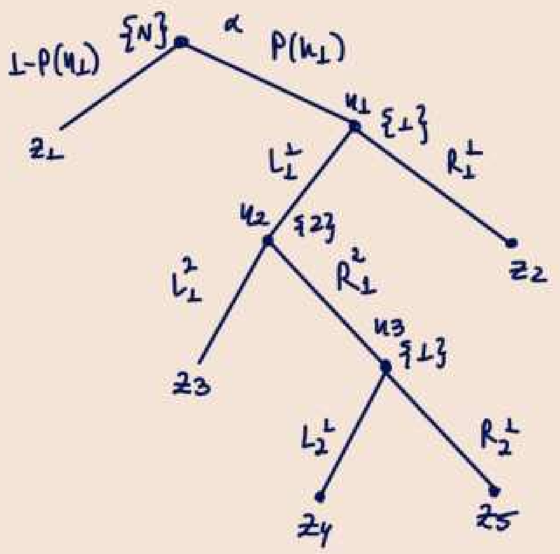
Formally we are interested in knowing if, given $\gamma^{-i} \in \Gamma^{-i}$,

$$\max_{\beta^i \in B^i} u^i(\beta^i, \gamma^{-i}) \stackrel{?}{=} \max_{\sigma^i \in \Sigma^i} u^i(\sigma^i, \gamma^{-i})$$

The LHS is the maximum utility the agent gets if he is allowed to choose a behavioural strategy, and the RHS is the max utility the agent gets if he is allowed to choose a mixed strategy, given that the other player follow γ^{-i}.

Definition: Given $\gamma \in \Gamma$, $Pr_\gamma(z) \in \Delta(Z_1)$ is the probability of ending in the final node $z \in Z_1$.

Example:



$Z = \{z_1, z_2, z_3, z_4, z_4\}$
$X = \{x_1, x_2, x_3\}$
$I_1^1 = \{x_1\}$ $I_2^1 = \{x_3\}$
$I_2^1 = \{x_2\}$
$C_1^1 = \{L_1^1, R_1^1\}$
$C_2^1 = \{L_2^1, R_2^1\}$
$C_1^2 = \{L_1^2, R_1^2\}$

In this example

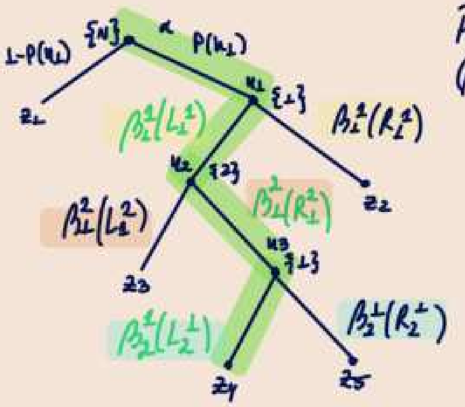
$$
\begin{aligned}
B^1 &= \Delta(C_1^1) \times \Delta(C_2^1) \\
&= \Delta(\{L_1^1, R_1^1\}) \times \Delta(\{L_2^1, R_2^1\}) \\
&= \left\{ (\beta_1^1(L_1^1), \beta_1^1(R_1^1), \beta_2^1(L_2^1), \beta_2^1(R_2^1)) \in [0,1]^4 : \right. \\
&\qquad \left. \beta_1^1(L_1^1) + \beta_1^1(R_1^1) = 1 \ \wedge \ \beta_2^1(L_2^1) + \beta_2^1(R_2^1) = 1 \right\}
\end{aligned}
$$

Take $\beta^1 \in B^1$, $\beta^2 \in B^2$, $\sigma^1 \in \Sigma^1$, $\sigma^2 \in \Sigma^2$.

$$
\begin{cases}
\beta^1 = (\beta_1^1(L_1^1), \beta_1^1(R_1^1), \beta_2^1(L_2^1), \beta_2^1(R_2^1)) \\
\beta^2 = (\beta_1^2(L_1^2), \beta_1^2(R_1^2)) \\
\sigma^1 = (\sigma^1(L_1^1, L_2^1), \sigma^1(L_1^1, R_2^1), \sigma^1(R_1^1, L_2^1), \sigma^1(R_1^1, R_2^1)) \\
\sigma^2 = (\sigma^2(L_1^2), \sigma^2(R_1^2))
\end{cases}
$$

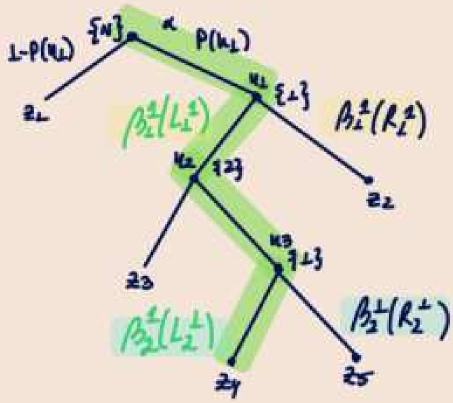
Take all possible combinations (β_1, β_2), (β_1, σ_2), (σ_1, σ_2), (σ_1, β_2) and compute the probability of reaching z_4.

CASE I: $(\beta 1, \beta 2)$



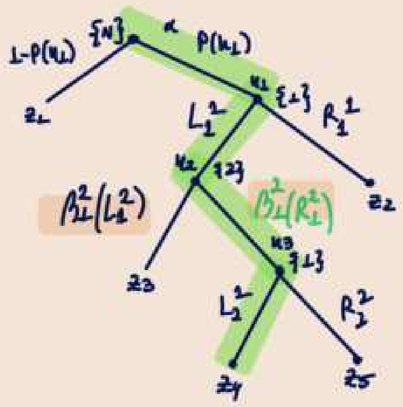
$$Pr(z_4) = P(u_1) \cdot \beta_1^1(L_1^1) \cdot \beta_2^2(R_1^2) \cdot \beta_2^1(L_2^1)$$
(β_1, β_2)

CASE II: $(\beta 1, \sigma^2)$



$$Pr(z_4) = P(u_1) \cdot \beta_1^1(L_1^1) \cdot \sigma_2^2(R_1^2) \cdot \beta_2^1(L_2^1)$$
(β_1, β_2)

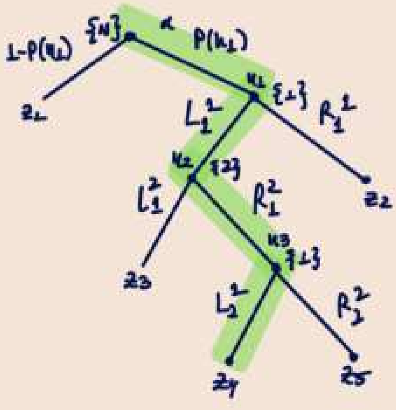
CASE II: (σ^1, β^2)



$$Pr(z_4) = P(u_1)\, \beta_1^2(R_1^2) \cdot \sigma^1(L_1^1, L_2^1)$$
(σ^1, β^2)

why not consider $\sigma^1(L_1^1, R_2^1)$?

CASE II: (σ^1, σ^2)

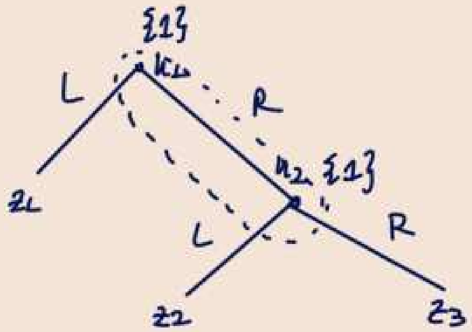


$$Pr(z_4) = P(u_1)\, \sigma^2(R_1^2) \cdot \sigma^1(L_1^1, L_2^1)$$
(σ^1, β^2)

Exercise: Assign numbers to $(\beta^1, \beta^2, \sigma^2, \sigma 2)$ and repeat this steps!

Question: Is there any difference between the set of prob. distributions over Z_1 generated by using $\gamma \in B$ or $\gamma \in \Sigma$? YES.

Example:



$Z_1 = \{z_1\ z_2\ z_3\}$

$X_1 = \{n_1, n_2\}$

$I_1^1 = \{n_1, n_2\}$

$C_1^1(I_1^1) = \{L, R\}$.

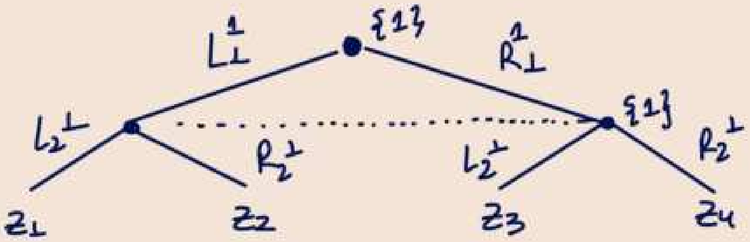
Take $\beta^1 = (0.5, 0.5)$. $Pr_{\beta^1}(z_2) = 0.25$ $Pr_{\beta^1}(z_1) = 0.5$ $Pr_{\beta_1}(z_3) = 0.25$.
Is there any way of getting this probability distribution over final nodes using $\sigma^1 \in \Sigma^1$?

Let $\sigma^1 = (\sigma^1(L), \sigma^1(R)) \in [0,1]^2$ and $\sigma^1(L) + \sigma^1(R) = 1$. $Pr_{\sigma_1}(z_2) = 0$ $\forall\ \sigma^1 \in \Sigma^1$.

Intuition: If the agent plays $\beta^1 = (0.5, 0.5)$ at every node in the information set, he will randomize between L or R with probability 1/2. If the agent plays $\sigma^1 = (0.5, 0.5)$, before the game starts, he will choose to play L or R (at every node in the information set) with probability 1/2.

Result: For this game the set of prob. distributions is bigger under B than under Σ. Is it allway the case? No.

Example:



Take $\sigma^1 = \left(\sigma^1(L_1^1, L_2^1),\ \sigma^1(L_1^1, R_2^1),\ \sigma^1(R_1^1, L_2^1),\ \sigma^1(R_1^1, R_2^1)\right)$
$= (1/2,\ 0,\ 0,\ 1/2).$

$$\Pr_{\sigma^1}(z) = \begin{cases} 1/2 & z \in \{z_1, z_4\} \\ 0 & z \in \{z_2, z_3\} \end{cases}$$

Can we obtain this probabilities using B^1? Note that for any $\beta^1 \in B^1$

$$\Pr_{\beta_1}(z_1) = \beta_1^1(L_1^1) \cdot \beta_2^1(L_2^1)$$
$$\Pr_{\beta_1}(z_2) = \beta_1^1(L_1^1) \cdot (1 - \beta_2^1(L_2^1))$$
$$\Pr_{\beta_1}(z_3) = \beta_1^1(R_1^1) \quad \beta_2^1(L_2^1)$$
$$\Pr_{\beta_1}(z_4) = \beta_1^1(R_1^1) \quad (1 - \beta_2^1(L_2^1))$$

If $\Pr_{\beta_1}(z_1) = \frac{1}{2}$ then $\beta_1^1(L_1^1) > 0$ and $\beta_2^1(L_2^1) > 0$.
If $\Pr_{\beta_1}(z_4) = \frac{1}{2}$ then $\beta_1^1(R_1^1) > 0$ and $\beta_2^1(L_2^1) < 1$.
$\Rightarrow \Pr_{\beta_1}(z_2) > 0 \quad \forall\ \beta_1 \in B^1.$

"Equivalent"

Definition: Take γ^{-i} as given. We say $\gamma^i \sim \tilde{\gamma}^i$ against γ^{-i} if:

$$\Pr_{(\gamma^i, \gamma^{-i})}(z) = \Pr_{(\tilde{\gamma}^i; \gamma^{-i})}(z) \quad \forall \ z \in Z_1.$$

Proposition: A game of perfect recall is a linear game.

Go to the previous recitation if you don't remember the definitions!

Proof: Suppose the game is not linear, then $\exists \ i \in \{1, \dots, I\}$ and I_k^i $k \in \{1, \dots, \# I^i\}$, $z \in Z_1$ such that:

$$\# \{ Path(z) \cap I_k^i \} > 1.$$

Take any $x, y \in (path(z) \cap I_k^i)$. Since the path to z is a totally ordered set; either $x > y$ or $y > x$, but this cannot happen in a game of perfect recall.

Theorem: In linear games, for every γ^{-i} and every $\beta^i \in B^i$ there exists $\sigma^i \in \Sigma^i$, such that: $\Pr_{(\beta^i, \gamma^{-i})}(z) = \Pr_{(\sigma^i, \gamma^{-i})}(z) \quad \forall \ z \in Z_1$.

Proof: See below for the case where $\gamma^{-i} \in \bigtimes_{j=1}^{I-i} S^j$. (We assume all other players play pure strategies).

Proof: Given a behavioral strategy we need to construct an equivalent mixed strategy strategy.

$\forall i$.

Let $\beta^i(A^i(I_k^i))$ denote the probability distribution over pure actions at infon set I_k^i under behavioral strategy β^i

$\beta^i(a^i(I_k^i))$ is the prob attached to action a^i at I_k^i under β^i.

$s^i(I_k^i)$ denote the action taken at infon set I_k^i under pure strategy s^i.

A mixed strategy is a probability disbn over pure strategy

Hence, define:

$$\sigma^i_{\beta^i}(s^i) = \prod_{j=1}^{I_k^i} \beta_j^i(s^i(I_j^i)) \quad \text{---(1)}$$

(i.e. prob assigned to <u>pure strategy</u> s^i is the product of probabilities assigned to <u>actions</u> corresponding to the strategy under behavioral strategy)

Now we'll show that mixed strategy constructed from behavioral strategy as shown above induces the same probability distbⁿ on the final nodes

If a final node z is always reached or never reached independent of the action of player i, then $\beta^i \sim \sigma^i$ trivially

Let $\tilde{a}^i_{I_k^i}(z)$ denote the action of player i at info set I_k^i leading to final node z

$$\tilde{I}^i(z) = \{ I_k^i \mid \text{path}(z) \cap I_k^i \neq \phi \}$$

$\tilde{I}^i(z)$ is the set of infoⁿ set that crosses path to z

Then

$$\mathbb{P}r_{\beta^i}(z) = \prod_{\tilde{I}_k^i(z)} \beta_j^i \left(\tilde{a}^i_{I_k^i}(z) \right)$$

(Prob on z is the product of probabilities on actions leading to z)

(note that $\mathbb{P}r_{\beta^i}(z)$ will depend on the strategy of other players. we are taking the strategies of other players as given and hence using the notation as shown above)

Now let us look at the probability induced by $\sigma^i_{\beta^i}$ on node z.

Let $\tilde{S}^i(z)$ be the set of pure strategies leading to z.

It must be the case that

$\forall s^i \in \tilde{S}^i(z)$,

By linearity

$$s^i(I^i_k) = \tilde{a}^i_{I^i_k}(z) \qquad (2)$$

$$\forall I^i_k \in \tilde{I}^i$$

i.e at each information set which crosses path to z, the strategy must select the action leading to z.

Then,

$$Pr_{\sigma^i_{\beta^i}}(z) = \sum_{s^i \in \tilde{S}(z)} \sigma^i_{\beta^i}(s^i)$$

$$= \sum_{s^i \in \tilde{S}(z)} \prod_{j=1}^{I^i_k} \beta^i_j(s^i(I^i_j))$$

(from (1))

$$= \sum_{s^i \in \tilde{S}(z)} \prod_{\tilde{I}^i(z)} \beta_j^i\left(s^i(I_k^i)\right) \prod_{I^i \setminus \tilde{I}^i(z)} \beta_j^i\left(s^i(I_k^i)\right)$$

$$= \sum_{s^i \in \tilde{S}(z)} \underbrace{\prod_{\tilde{I}^i(z)} \beta_j^i\left(\tilde{a}^i(I_k^i)\right)} \prod_{I \setminus \tilde{I}^i(z)} \beta_j^i\left(s^i(I_k^i)\right)$$

(from (2))

This component remains fixed over $\tilde{S}(z)$
so, we can take it outside the summation.

$$= \prod_{\tilde{I}^i(z)} \beta_j^i\left(\tilde{a}^i(I_k^i)\right) \underbrace{\sum_{s^i \in \tilde{S}(z)} \prod_{I \setminus \tilde{I}^i(z)} \beta_j^i\left(s^i(I_k^i)\right)}$$

Note that all these probabilities will add upto 1.

$$= \prod_{\tilde{I}^i(z)} \beta_j^i\left(\tilde{a}^i(I_k^i)\right)$$

$$= \Pr_{\beta^i}(z)$$

To see that the term; $\sum_{s^i \in \tilde{S}(z)} \prod_{I^i \setminus \tilde{I}^i(z)} \beta_j^i(s^i(I_k^i)) = 1$

, we can simply interchange the sum and product to get

$$\prod_{I^i \setminus \tilde{I}^i(z)} \sum_{s^i \in \tilde{S}(z)} \beta_j^i(s^i(I_k^i))$$

Now $\forall \; I_k^i \in I^i \setminus \tilde{I}^i(z)$

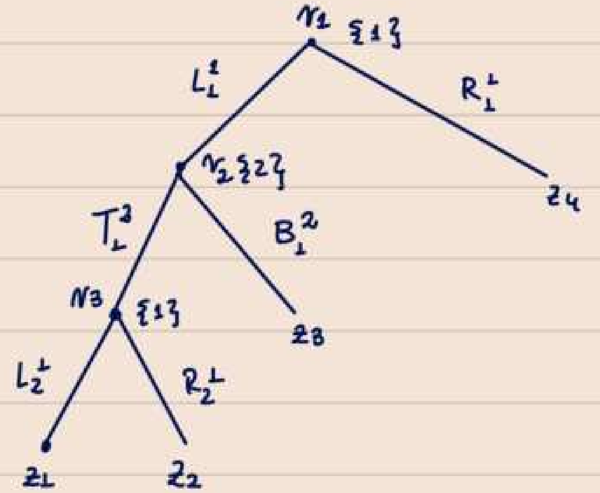
$$\sum_{s^i} \beta_j^i(s^i(I_k^i)) = 1,$$

This is because in paths not leading to z', $\tau(z)$ will contain all possible choice of actions at $I_k^i \in I^i \setminus \tilde{I}^i(z)$, which add upto 1 since β^i is a probability dist^n over those actions.

$\blacksquare$

Now starting with σ we want to obtain the equivalent behavioural strategy:

Example:



$I_1^1 = \{v_1\}$
$I_2^1 = \{v_3\}$
$I_1^2 = \{v_2\}$

take $\sigma^1 = (\sigma^1(L_1^1, L_2^1), \sigma^1(L_1^1, R_2^1), \sigma^1(R_1^1, L_2^1), \sigma^1(R_1^1, R_2^1))$

Note that we need to know what player 2 is doing in order to compute the probability distribution over the final nodes, given an action of player 1.

Definition: For $\forall i$, $\forall (I_j^i)$ the relevant set:

$$Rel(I_j^i) \equiv \{ s^i \in S^i : \exists_{s^{-i} \in S^{-i}} \ path(s^i, s^{-i}) \cap I_j^i \neq \emptyset \}$$

where $Rel(I_j^i) \subseteq S^i$.

why (R_1^1, L_2^1) (R_1^1, R_2^1) do not belong here?

Example: $Rel(I_2^1) = \{ (L_1^1; L_2^1), (L_1^1, R_2^1) \}$
$Rel(I_1^1) = S^1 \quad Rel(I_2^1) = S^2$.

Intuition: If $s^i \in Rel(I_j^i)$, there exists a strategy for other players s^{-i} such that if agents follow (s^i, s^{-i}) the path generated by following that strategy profile intersects the information set (I_j^i).

The general algorithm for player i:

$$\beta^i_j(c^i_j; \sigma^i) = \frac{\sum_{\{s^i \in Rel(I^i_j) : s^i(I^i_j) = c^i_j\}} \sigma^i(s^i)}{\sum_{\{s^i \in Rel(I^i_j)\}} \sigma^i(s^i)}$$

for $c^i_j \in C(I^i_j)$

$$= \text{arbitrary if } \sum_{\{s^i \in Rel(I^i_j)\}} \sigma^i(s^i) = 0$$

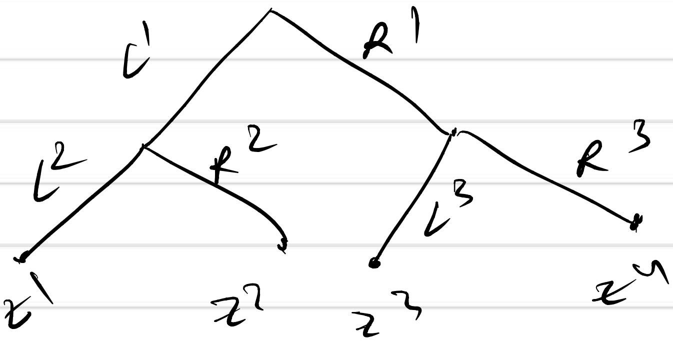
$\blacksquare$

Exercise: Draw a simple game of not perfect recall (for example the one we covered in the previous notes) see why the algorithm fails.

<u>Kuhn Theorem</u>: In games of perfect recall $B \gtrsim \Sigma$

(i.e. $\forall \sigma^i \in \Sigma^i$, $\exists \beta^i \in B^i$, s.t. $\sigma^i \sim \beta^i$)

Let us look at the construction of behavioural strategy from a mixed strategy with help of following example.



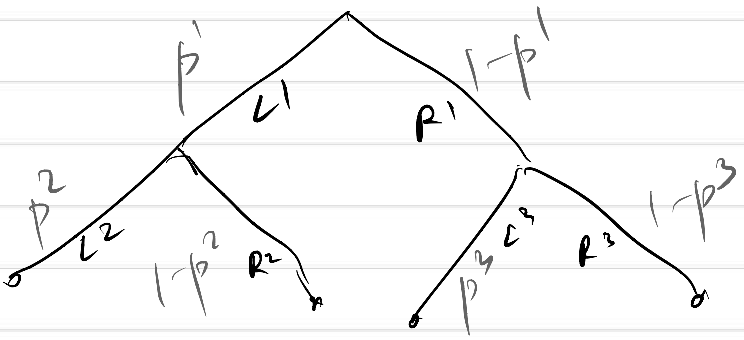
Pure St:

$$S = \{ L^1L^2L^3,\ L^1L^2R^3 \ \ldots\ldots \}$$

Mixed St

$$\Sigma = \Delta(S)$$

Behavioural St:



Given $\sigma \in \Sigma$, we need to find

$\beta = (p^1, p^2, p^3)$ s.t $\beta \sim \sigma$.

Define

$$\sigma(L^1) = \sigma(L^1 L^2 L^3) + \sigma(L^1 L^2 R^3) + \sigma(L^1 R^2 L^3) + \sigma(L^1 R^2 R^3)$$

construct

$$p^1 = \sigma(L^1)$$

Similarly define

$$\sigma(L^1 L^2) = \sigma(L^1 L^2 L^3) + \sigma(L^1 L^2 R^3)$$

Construct $p^2 = \dfrac{\sigma(L^1 L^2)}{\sigma(L^1)}$

Now note that

$$\mathbb{P}r_B(z^1) = p^1 p^2 = \sigma(L^1 L^2)$$

$$\mathbb{P}r_\sigma(z^1) = \sigma(L^1 L^2)$$

which is what we need.

The above construction show that, we can define the probability distⁿ over actions at information set as the distbⁿ given by mixed strategy condⁿ on reaching the information set.

$B^i_{\sigma^i}(a^i(I^i_j) \mid I^i_j)$ is the probability to action a^i at infoⁿ set I^i_j

$$B^i_{\sigma^i}(a^i(I^i_j) \mid I^i_j) = \frac{\mathbb{P}r_\sigma(\{\text{reach } I^i_j\} \cap \{\text{choose } a^i\})}{\mathbb{P}r_\sigma(\{\text{reach}(I^i_j)\})}$$

where $\mathbb{P}r_\sigma(\{\text{reach } x(I^i_k)\})$ can be written more clearly based on the following idea.

Note that a path to $x(I^i_k)$ will consists of actions taken by agents in previous infoⁿ set leading to node $x(I^i_k)$.

Let $a^m_{x(I^i_k)}(I^m_j)$ denote the action by agent m at I^m_j that leads to $x(I^i_k)$.

Let $\mathfrak{Z}^m_{x(I^i_k)}$ be the set of pure st. for agent m that selects $a^m_{x(I^i_k)}(I^m_j)$ at I^m_j

Then

$$\mathbb{P}r_{\sigma}(\{\text{reach } x(I_k^i)\}) = \prod_{i \in I} \sum_{s^i \in \tilde{S}^i_{x(I_k^i)}} \sigma^i(s^i)$$

Let $\hat{S}^i_{x(I_k^i)}$ denote the set of st which selects action $a^i_{a(I_k^i)}(I_j^i)$ and $a^i(I_k^i)$.

Then;

$$\mathbb{P}r_{\sigma}(\{\text{choose } a^i(I_k^i)\} \cap \{\text{reach } I_k^i\})$$

$$= \left(\sum_{s^i \in \hat{S}^i_{x(I_k^i)}} \sigma^i(s^i)\right) \sum_{x \in I_k^i} \prod_{j \in I \setminus \{i\}} \sum_{s^j \in \tilde{S}^j_{x(I_k^i)}} \sigma^j(s^j)$$

Thus;

$$\beta^i(a^i(I_k^i)) = \frac{\left(\sum_{s^i \in \hat{S}^i_{x(I_k^i)}} \sigma^i(s^i)\right) \sum_{x \in I_k^i} \prod_{j \in I \setminus \{i\}} \sum_{s^j \in \tilde{S}^j_{x(I_k^i)}} \sigma^j(s^j)}{\sum_{x \in I_k^i} \prod_{i \in I} \sum_{s^i \in \tilde{S}^i_{x(I_k^i)}} \sigma^i(s^i)}$$

The above is a general way to write the probabilities, however we can simplify the expression noting that the sum of prob due to the st of other players remains fixed in the numerator and denominator.

Taking $\sum_{x \in I_k^i} \prod_{j \in I \setminus \{i\}} \sum \sigma^j(s^j) \neq 0$ out common, we have:

$$\beta^i(a^i(I_k^i)) = \frac{\sum_{s^i \in \hat{S}^i_x(I_k^i)} \sigma^i(s^i)}{\sum_{x \in I_k^i} \sum_{s^i \in \tilde{S}^i_{x}(I_k^i)} \sigma^i(s^i)}$$

<u>Note</u>: writing the prob as above requires $\sum_{x \in I_k^i} \prod_{j \in I \setminus \{i\}} \sum \sigma^j(s^j) \neq 0$ i.e under the strategies of other players there must be positive prob on reaching the infor set.

However a more general idea is to assign the same prob even when $\sum_{x \in I_k^i} \prod_{j \in I \setminus \{i\}} \sum \sigma^j(s^j) = 0$.

#

Following Prof. Aldo's notes, the above can also be written in terms of Relevant Strategies:

$$Rel(I_k^i) = \{ s^i \in S^i : \exists s^{-i} \in S^{-i} \text{ s.t. } Path(s^i, s^{-i}) \cap I_k^i \neq \phi \}$$

Then

$$\beta^i(a^i(I_k^i)) = \frac{\sum_{\{s^i \in Rel(I_k^i);\ s^i(I_k^i) = a^i\}} \sigma^i(s^i)}{\sum_{\{s^i \in Rel(I_k^i)\}} \sigma^i(s^i)}$$