8101 - Recitation II: Producer theory and Consumer theory.

Producer Theory

- **Definition**: For N points $\{x_1, ..., x_N\}$, the convex hull C is then given by the expression:
$$C = \left\{ \sum_{j=1}^{n} \lambda_j x_j : \lambda_j \geq 0 \quad \forall j \quad and \quad \sum_{j=1}^{N} \lambda_j = 1 \right\}.$$
- **Definition**: The (maximum) *profit* is:
$$\pi^* (p) := \sup_{y \in Y} py.$$
- **Definition:** The *supply* of firm at p:
$$s^* (p) = \{y^* \in Y : py^* \geq py, \quad \forall y \in Y\}.$$
- **Definition**: For a closed set $K \in R^L$, *the support function* μ_K is defined by:
$$\mu_K (p) = \sup_{x \in K} px$$
for every $p \in R^L$.
- **Corollary** Rockafellar (1970): For a closed and convex set $K \in R^L$:
$$K = \left\{ x \in R^L : px \leq \mu_K (p), \quad \forall p \right\}$$
- **Definition**: The set Y_π that *profit-rationalizes* function π is:
$$Y = \left\{ y \in R^L : py \leq \pi (p) \right\}.$$

Consumer Theory

- **Definition**: The *consumption set* is a subset $X \subseteq R^L$.
- **Definition**: $x \in X$ is a *consumption bundle.*
- **Definition**: $\succeq$ is the *preference relation* over commodity bundles in X.
- **Remark**: Preference relation $\succeq$ induces strict preference $\succ$ and indifference $\sim$.
- **Properties** of the preference relation:
  - *Transitive*: For all $x, y, z \in X$ if $x \succeq y$ and $y \succeq z$ then $x \succeq z$.
  - *Completeness*: For all $x, y \in X$ we have $x \succeq y$ or $y \succeq x$ or both.
  - *Continuous*:
    - Defintion I: The preference relation $\succeq$ on X is continuous if for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succeq y^n$ for all n, $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$, we have that $x \succeq y$.
    - Definition II: for all x, the upper contour set $\{y \in X : y \succeq x\}$ and the lower contour set $\{y \in X : x \succeq y\}$ are both closed.
  - *Locally nonsatiated*: For every $x \in X$ and every $\epsilon > 0$, there is $y \in X$ such that $\|y - x\| \leq \epsilon$ and $y \succ x$.
  - *Weakly monotone - Monotone* (Increasing): If $x \in X$ and $y >> x$ implies $y \succ x$.
  - *Strongly monotone* (Strictly increasing):If $x \in X$ and $y \geq x$ for $x \neq y$ implies $y \succ x$.
  - *Convex*: If for every $x \in X$, the upper contour set $\{y \in X : y \succeq x\}$ is convex, that is, for any $y \succeq x$ and $z \succeq x$ then $\alpha z + (1-\alpha)y \succeq x$ for any $\alpha \in [0, 1]$.
  - *Strictly convex*: If for any $y \succeq x$ and $z \succeq x$ such that $y \neq z$ then $\alpha z + (1-\alpha)y \succ x$ for any $\alpha \in (0, 1)$.
  - *Homothethic*: A monotone preference relation $\succeq$ on $X = R^L$ is homothetic if $x \sim y$, then $\lambda x \sim \lambda y$ for any $\lambda > 0$.
  - *Quasilinear*: The preference relation $\succsim$ on $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is quasilinear with respect to commodity 1 (called, in this case, the numeraire commodity) if:
    - All the indifference sets are parallel displacements of each other along the axis of commodity 1 . If $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \ldots, 0)$ and any $\alpha \in \mathbb{R}$.
    - Good 1 is desirable; that is, $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.
- **Example**: Lexicographic preference relation $\succeq$. Assume $X = R^2$. Define $x \succeq y$ if either $x_1 > y_1$ or if $x_1 = y_1$ and $x_2 \geq y_2$.
- **Exercise:** One can show this preference ralation is complete, transitive, strongly monotone and strictly convex but not continuous.

Old problem set question

Consider two closed and convex production sets $Y_1, Y_2 \subset R^L$ such that $0 \in Y_1$ and $0 \in Y_2$. Let π_i^* and s_i^* denote the profit function and the supply associated with Y_i, for $i = 1, 2$. Prove that $Y_1 \subset Y_2$ if and only if $\pi_1^*(p) \leq \pi_2^*(p)$ for every $p \in R^L$.

Prelim 2010 Spring.

Let $Y \subset \mathbb{R}^L$ be a finite production set, i.e., a set consisting of a finite number of production plans,$Y = \{y^1, y^2, \ldots, y^n\}$. Let $\hat{Y}$ be the convex hull of Y, that is the set of all convex combinations of production plans $y^1, \ldots, y^n$. Further, let $\pi_Y^*, s_Y^*, \pi_{\hat{Y}}^*$, and $s_{\hat{Y}}^*$ be the (maximum) profit functions and the supply correspondences for Y and $\hat{Y}$, respectively.

Show that $\pi_Y^* = \pi_{\hat{Y}}^*$ for every $p \in \mathbb{R}^L$.

1. Show that $s_Y^* \subseteq s_{\hat{Y}}^*$ for every $p \in \mathbb{R}^L$. Show that the inclusion is strict for some price vectors if $n \geq 2$.

Prelim 2008 Fall.

Consider a finite set of observations $\{p^t, y^t\}_{t=1}^T$ of pairs of price vectors $p^t \in \mathbb{R}_{++}^l$ and production plans $y^t \in \mathbb{R}^l$. We say that production set $Y \subset \mathbb{R}^l$ profit-rationalizes this set of observations if $y^t \in Y$ and $p^t y^t = \max_{y \in Y} p^t y$ for every t.

1. State the weak axiom of profit maximization (WAPM).
2. Prove that a set of observations $(p^1, y^1), \ldots, (p^T, y^T)$ satisfies WAPM if and only if there exists a closed, convex production set Y that profit-rationalizes these observations.

Prelim 2007 Spring.

Let $\succeq$ be a transitive and complete preference relation on the consumption set $X = \mathbb{R}_+^L$. Consider two conditions often used as alternative definitions of continuity of $\succeq$:

- $C1$: The preference relation $\succeq$ on X is continuous if for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^\infty$ with $x^n \succeq y^n$ for all n, $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$, we have that $x \succeq y$.
- $C2$: For all x, the upper contour set $\{y \in X : y \succeq x\}$ and the lower contour set $\{y \in X : x \succeq y\}$ are both closed.

Prove that conditions C_1 and C_2 are equivalent.

Prelim 2007 Fall.

Let $\succeq$ be a reflexive, transitive, complete, and strictly increasing (i.e., strongly monotone) preference relation on the consumption set $X = \mathbb{R}_+^L$. Preference relation $\succeq$ is said to be homothetic if the following holds for every $x, x' \in \mathbb{R}_+^L$ and every $\lambda > 0$: if $x \sim x'$ then$\lambda x \sim \lambda x'$ where $\sim$ is the indifference relation of $\succeq$. Prove that $\succeq$ is homothetic if and only if there exists a utility representation u of $\succeq$ such that u is homogeneous of degree 1.