8101 - Recitation I: Production sets & Supermodularity.

Production sets and production functions

- **Definition**: A *production set* Y is a subset of the comodity space $\mathbb{R}^L$, where L is the number of commodities.
- **Definition**: A *production plan* is $y \in Y$.
- Some properties of the production sets:
  - Y is nonempty: $Y \neq \emptyset$.
  - Y is closed: $y^n \to y$ and $y^n \in Y$, then implies $y \in Y$.
  - No free lunch: $Y \cap \mathbb{R}^L_+ \subset \{0\}$.
  - Innaction: $0 \in Y$.
  - Free disposal: $Y - \mathbb{R}^L_+ \subset Y$.
  - Irreversability: Take any $y \in Y$ with $y \neq 0$, then $-y \in Y$.
  - Nonincreasing returns to scale (NIRS): Take any $y \in Y$, then $\lambda y \in Y$ for all $0 \leq \lambda \leq 1$.
  - Nondecreasing returns to scale (NDRS): Take any $y \in Y$, then $\lambda y \in Y$ for all $\lambda \geq 1$.
  - Constant returns to scale (CRS): Take any $y \in Y$, then $\lambda y \in Y$ for all $\lambda \geq 0$.
  - Additivity: Suppose $y, y' \in Y$, then $y + y' \in Y$.
  - Convexity: if y,$y' \in Y$, then $\forall \lambda \in [0,1]$, $\lambda y + (1-\lambda)y' \in Y$.
  - Convex cone: for any y,$y' \in Y$, and constants $\alpha \geq 0$ and $\beta \geq 0$, we have $\alpha y + \beta y' \in Y$.
- **Definition:** $f : \mathbb{R}^L_+ \to \mathbb{R}$ is a *production function.*
- Some properties of the production functions:
  - Constant returns to scale (CRS): $f(\lambda x) = \lambda f(x)$, for every $\lambda \geq 0$ and $x \geq 0$.
  - Decreasing returns to scale (DRS): $f(\lambda x) < \lambda f(x)$, for every $\lambda > 1$ and $x \geq 0$ such that $f(x) \neq 0$.
  - Increasing returs to scale (IRS): $f(\lambda x) > \lambda f(x)$, for every $\lambda > 1$ and $x \geq 0$ such that $f(x) \neq 0$.
  - Continuous, Differentiable, etc.
- **Remark:** A production function gives rise to a production set:

$$Y_f = \left\{(x, z) \in \mathbb{R}^{L+1}_+ : 0 \leq z \leq f(-x) \quad and \quad x \leq 0\right\}.$$

- **Exercise 1:** Show that constant, decreasing or increasing returns to scale for a production function f imply that the production set Y_f exhibits constant, nonincreasing or nondecreasing returns to scale, respectively.
- **Exercise 2:** Show that if Y is additive and exhibits CRS if and only if it is a convex cone.
- **Exercise 3:** Consider a closed and convex production set $Y \subset \mathbb{R}^L$ such that $\overrightarrow{0} \in Y$. Prove that the free disposal property holds iff $\mathbb{R}^{\mathbb{L}}_- \subset Y$.
- **Exercise 4:** Give and example of a non-convex but closed production set in $\mathbb{R}^2$ satisfying $\mathbb{R}^{\mathbb{L}}_- \subset Y$ but not free disposal property.
- **Exercise 5:** Give and example of a convex, closed and NIRS production set in $\mathbb{R}^2$ that violates no free lunch.
- **Exercise 6:** Consider the production set given by:

$$Y = \left\{(y_1, y_2, y_3) : y_1 \leq 0, y_2 \leq 0, y_3 \leq \min\left\{\sqrt{-y_1}, \sqrt{-2y_2}\right\}\right\}$$

  Verify whether Y exhibits nondecreasing returns to scale, nonincreasing returns to scale, or neither.

Supermodularity

- **Notation**: We use $x \vee y$ to denote the *supremum of* x *and* y element wise.
- **Notation**: We use $x \wedge y$ to denote the *infimum of* x *and* y element wise
- **Definition**: Function $f\colon \mathbb{R}_+^n \to \mathbb{R}$ is *supermodular* if for every $x, y \in \mathbb{R}_+^n$:

$$f\left(x \vee y\right) + f\left(x \wedge y\right) \geq f\left(x\right) + f\left(y\right).$$

- **Proposition**: Let $f\colon \mathbb{R}_+^n \to \mathbb{R}$ be twice differentiable. Then f is supermodular if and only if:

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0$$

  for every $i, j, i \neq j$ and every $x \in R_+^n$.

- **Exercise 7:** Show that the Leontief function $f\colon \mathbb{R}_+^2 \to \mathbb{R}$, where $f(x) = \min_i \{x_1, x_2\}$ is supermodular.
- **Exercise 8:** Show that the Leontief function $f\colon \mathbb{R}_+^n \to \mathbb{R}$, where $f(x) = \min_i \{\alpha_i x_i\}$ with $\alpha > 0$ is supermodular.
- **Exercise 9:** Give and example of a strictly increasing function that is not supermodular.
- **Exercise 10** : Show that if $f(x) : [0,1]^n \to \mathbb{R}$ is supermodular then $\hat{f}\left(x\right) \equiv f(1-x)$ is supermodular in $x \in [0,1]$.