8101 - Recitation I: Production sets & Supermodularity.

Production sets and production functions

- **Definition**: A *production set* Y is a subset of the comodity space $\mathbb{R}^L$, where L is the number of commodities.
- **Definition**: A *production plan* is $y \in Y$.
- Some properties of the production sets:
  - Y is nonempty: $Y \neq \emptyset$.
  - Y is closed: $y^n \to y$ and $y^n \in Y$, then implies $y \in Y$.
  - No free lunch: $Y \cap \mathbb{R}^L_+ \subset \{0\}$.
  - Innaction: $0 \in Y$.
  - Free disposal: $Y - \mathbb{R}^L_+ \subset Y$.
  - Irreversability: Take any $y \in Y$ with $y \neq 0$, then $-y \in Y$.
  - Nonincreasing returns to scale (NIRS): Take any $y \in Y$, then $\lambda y \in Y$ for all $0 \leq \lambda \leq 1$.
  - Nondecreasing returns to scale (NDRS): Take any $y \in Y$, then $\lambda y \in Y$ for all $\lambda \geq 1$.
  - Constant returns to scale (CRS): Take any $y \in Y$, then $\lambda y \in Y$ for all $\lambda \geq 0$.
  - Additivity: Suppose $y, y' \in Y$, then $y + y' \in Y$.
  - Convexity: if y,$y' \in Y$, then $\forall \lambda \in [0,1]$, $\lambda y + (1-\lambda)y' \in Y$.
  - Convex cone: for any y,$y' \in Y$, and constants $\alpha \geq 0$ and $\beta \geq 0$, we have $\alpha y + \beta y' \in Y$.
- **Definition:** $f : \mathbb{R}^L_+ \to \mathbb{R}$ is a *production function.*
- Some properties of the production functions:
  - Constant returns to scale (CRS): $f(\lambda x) = \lambda f(x)$, for every $\lambda \geq 0$ and $x \geq 0$.
  - Decreasing returns to scale (DRS): $f(\lambda x) < \lambda f(x)$, for every $\lambda > 1$ and $x \geq 0$ such that $f(x) \neq 0$.
  - Increasing returs to scale (IRS): $f(\lambda x) > \lambda f(x)$, for every $\lambda > 1$ and $x \geq 0$ such that $f(x) \neq 0$.
  - Continuous, Differentiable, etc.
- **Remark:** A production function gives rise to a production set:

$$Y_f = \left\{(x,z) \in \mathbb{R}^{L+1}_+ : 0 \leq z \leq f(-x) \quad and \quad x \leq 0\right\}.$$

- **Exercise 1:** Show that constant, decreasing or increasing returns to scale for a production function f imply that the production set Y_f exhibits constant, nonincreasing or nondecreasing returns to scale, respectively.
  - Suppose $f : \mathbb{R}^L_+ \to \mathbb{R}$ has DRS, $f(\overrightarrow{0}) = 0$ and that $f(x) > 0$ for all $x \neq \overrightarrow{0}$.
  - We want to show that Y_f has NIRS.
  - We need to show that for any $y = (x,z) \in Y_f$ where $(x,z) \in \mathbb{R}^{\mathbb{L}}_- \times \mathbb{R}_+$ and any $0 \leq \lambda \leq 1$, $\lambda y = (\lambda x, \lambda z) \in Y_f$.
  - Since $y = (x,z) \in Y_f$, we know that:

$$0 \leq z \leq f(-x) \quad and \quad x \leq 0. \tag{0.1}$$

- We want to show that $\lambda y = (\lambda x, \lambda z) \in Y_f$ therefore we need to show that:
$$0 \leq \lambda z \leq f(-\lambda x) \quad and \quad \lambda x \leq 0.$$
- Multiply equation (0.1) by $\lambda \in (0,1)$:
$$0 \leq \lambda z \leq \lambda f(-x) \quad and \quad \lambda x \leq 0. \tag{0.2}$$
- Define $\hat{\lambda} = 1/\lambda$ and $\hat{x} = \lambda x$, note that $\hat{\lambda} > 1$.
- Since f has DRS we know that $f\left(-\hat{\lambda}\hat{x}\right) < \hat{\lambda} f\left(-\hat{x}\right)$.
- This implies $\lambda f(-x) < f(-\lambda x)$ for $\lambda \in (0,1)$.
- Therefore, using equation (0.2) holds we can show that $\lambda y \in Y_f$.

- **Exercise 2:** Show that if Y is additive and exhibits CRS if and only if it is a convex cone.
  - [$\Rightarrow$] Take any $y \in Y$. By CRS we know that $\alpha y \in Y$ for any $\alpha \geq 0$. Since additivity holds, we are done.
  - [$\Leftarrow$] Pick any y,$y' \in Y$. Choosing $\alpha = 1$ and $\beta = 1$ makes additivity hold. For CRS we just choose $\beta = 0$.

- **Exercise 3:** Consider a closed and convex production set $Y \subset \mathbb{R}^L$ such that $\overrightarrow{0} \in Y$. Prove that the free disposal property holds iff $\mathbb{R}^{\mathbb{L}}_{-} \subset Y$.
  - [$\Rightarrow$] If free disposal holds, and $0 \in Y$,then for any $x \in \mathbb{R}^{\mathbb{L}}_{-}$, we can write it as $x = 0 - (-x) \in Y$.
  - [$\Leftarrow$] Take any $y \in Y$ and any $x \in R^L_+$, we want to show that$(y - x) \in Y$ if $\mathbb{R}^{\mathbb{L}}_{-} \subset Y$. Note that $-x \in R^L_-$ and therefore for any natural number n,$-nx \in R^L_- \subset Y$. Note that by convexity:
  $$\left[\frac{1}{n}\right](-nx) + \left[1 - \frac{1}{n}\right] y \in Y$$
  Since Y is closed we know that the limit point $(y - x) \in Y$.

- **Exercise 4:** Give and example of a non-convex but closed production set in $\mathbb{R}^2$ satisfying $\mathbb{R}^{\mathbb{L}}_{-} \subset Y$ but not free disposal property.

- **Exercise 5:** Give and example of a convex, closed and NIRS production set in $\mathbb{R}^2$ that violates no free lunch.

- **Exercise 6:** Consider the production set given by:
$$Y = \left\{(y_1, y_2, y_3) : y_1 \leq 0, y_2 \leq 0, y_3 \leq \min\left\{\sqrt{-y_1}, \sqrt{-2y_2}\right\}\right\}$$
Verify whether Y exhibits nondecreasing returns to scale, nonincreasing returns to scale, or neither.
  - To show that the production set does not exhibit NDRS, try using $(-16, -8, 4)$ and $\lambda = 4$. You will find that $(-64, -32, 16) \notin Y$.
  - To show that the production set does exhibit NIRS, we want to show that for any $y \in Y$,and for any $0 < \lambda < 1$we have that $\lambda y \in Y$.
  - For that we need to show only that:
  $$\lambda y_3 \leq \min\left\{\sqrt{-\lambda y_1}, \sqrt{-2\lambda y_2}\right\}$$
  - Since $y \in Y$,and $\lambda \in (0,1)$:
  $$\begin{aligned} y_3 &\leq \min\left\{\sqrt{-y_1}, \sqrt{-2y_2}\right\} \\ \lambda y_3 &\leq \lambda \min\left\{\sqrt{-y_1}, \sqrt{-2y_2}\right\} \\ &\leq \sqrt{\lambda} \min\left\{\sqrt{-y_1}, \sqrt{-2y_2}\right\} \\ &= \min\left\{\sqrt{\lambda}\sqrt{-y_1}, \sqrt{\lambda}\sqrt{-2y_2}\right\} \\ &= \min\left\{\sqrt{-\lambda y_1}, \sqrt{-2\lambda y_2}\right\} \end{aligned}$$

Supermodularity

- **Notation**: We use $x \vee y$ to denote the *supremum of x and y* element wise.
- **Notation**: We use $x \wedge y$ to denote the *infimum of x and y* element wise
- **Definition**: Function $f\colon \mathbb{R}_+^n \to \mathbb{R}$ is *supermodular* if for every $x, y \in \mathbb{R}_+^n$:

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).$$

- **Proposition**: Let $f : \mathbb{R}_+^n \to \mathbb{R}$ be twice differentiable. Then f is supermodular if and only if:

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0$$

  for every $i, j, i \neq j$ and every $x \in R_+^n$.

- **Exercise 7:** Show that the Leontief function $f\colon \mathbb{R}_+^2 \to \mathbb{R}$, where $f(x) = \min_i \{x_1, x_2\}$ is supermodular.
  - Take any $x = (x_1, x_2)$ and $y = (y_1, y_2)$.
  - Claim 1: $f(x \vee y) = \min\{\max\{x_1, y_1\}, \max\{x_2, y_2\}\} \geq \max\{f(x), f(y)\}$.
  - Claim 2: $f(x \wedge y) = \min\{min\{x_1, y_1\}, min\{x_2, y_2\}\} = \min\{x_1, y_1, x_2, y_2\} = \min\{f(x), f(y)\}$
- **Exercise 8:** Show that the Leontief function $f\colon \mathbb{R}_+^n \to \mathbb{R}$, where $f(x) = \min_i \{\alpha_i x_i\}$ with $\alpha > 0$ is supermodular.
  - Claim: $f(x \vee y) \geq \max\{f(x), f(y)\}$.
  - Proof: (The same logic applies to $f(y)$):

$$\begin{aligned} f(x \vee y) &= \min_i \{\alpha_i \max\{x_i, y_i\}\} \\ &\geq \min_i \{\alpha_i x_i\} = f(x) \end{aligned}$$

  - Claim: $f(x \wedge y) = min\{f(x), f(y)\}$
  - Proof: (The same logic applies to $f(y)$). Note that:

$$f(x \wedge y) = \min_i \{\alpha_i \min\{x_i, y_i\}\}$$

    wlog assume $f(x \wedge y) = \alpha_j x_j$ for some j.
  - Note that we know $f(x) = \alpha_j x_j$. Why? Suppose not.
  - Suppose $f(x) = \alpha_k x_k$ for some k with $\alpha_k x_k \neq \alpha_j x_j$.
  - If $\alpha_k x_k < \alpha_j x_j$ this is a contradiction since $\alpha_j x_j \leq min\{\alpha_k x_k, \alpha_k y_k\}$. If $\alpha_k x_k > \alpha_j x_j$ this is also a contradiction.
- **Exercise 9:** Give and example of a strictly increasing function in ${\mathbb{R}_+}^2 \to \mathbb{R}$ that is not supermodular.
  - $f(x, y) = ln(x + y)$
- **Exercise 10**: Show that if $f(x) : [0,1]^n \to \mathbb{R}$ is supermodular then $\hat{f}(x) \equiv f(\overrightarrow{1} - x)$ is supermodular in $x \in [0,1]$. Note: $\overrightarrow{1} = (1, ..., 1)$.
  - Since $f(x)$ is supermodular for any $x \in [0,1]^n$:

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$$

– We want to show that for any $x, y \in [0,1]^n$:

$$\hat{f}(x \vee y) + \hat{f}(x \wedge y) \geq \hat{f}(x) + \hat{f}(y)$$

which is equivalent to showing that:

$$f\left(\overrightarrow{1} - (x \vee y)\right) + f\left(\overrightarrow{1} - (x \wedge y)\right) \geq f(\overrightarrow{1} - x) + f\left(\overrightarrow{1} - y\right).$$

– Lets rewrite the LHS. By Definition:

$$f\left(\overrightarrow{1} - (x \vee y)\right) + f\left(\overrightarrow{1} - (x \wedge y)\right) =$$

$$f\left(\overrightarrow{1} - (max\{x_1, y_1\}, ..., max\{x_n, y_n\})\right) + f\left(\overrightarrow{1} - (min\{x_1, y_1\}, ..., min\{x_n, y_n\})\right) =$$

$$f((min\{1-x_1, 1-y_1\}, ..., min\{1-x_n, 1-y_n\})) + f((max\{1-x_1, 1-y_1\}, ..., max\{1-x_n, 1-y_n\})) =$$

– Rename variables $\hat{x} = \overrightarrow{1} - x$ and $\hat{y} = \overrightarrow{1} - y$, where $\hat{x} = (1-x_1, ..., 1-x_n)$. Then

$$f((min\{\hat{x}_1, \hat{y}_1\}, ..., min\{\hat{x}_n, \hat{y}_n\})) + f((max\{\hat{x}_1, \hat{y}_1\}, ..., max\{\hat{x}_n, \hat{y}_n\})) =$$

$$f(\hat{x} \wedge \hat{y}) + f(\hat{x} \vee \hat{y})$$

– We then conclude that:

$$\hat{f}(x \vee y) + \hat{f}(x \wedge y) = f(\hat{x} \wedge \hat{y}) + f(\hat{x} \vee \hat{y})$$

– Since $\hat{x}, \hat{y} \in [0,1]^n$ and f is supermodular:

$$f(\hat{x} \vee \hat{y}) + f(\hat{x} \wedge \hat{y}) \geq f(\hat{x}) + f(\hat{y})$$

– Since:

$$f(\hat{x}) + f(\hat{y}) = f(1-x) + f(1-y) = \hat{f}(x) + \hat{f}(y)$$

we have proven that:

$$\hat{f}(x \vee y) + \hat{f}(x \wedge y) \geq \hat{f}(x) + \hat{f}(y).$$